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Quelques modèles intégrables dans la classe d'universalité KPZ

Some integrable models in the KPZ universality class

Sous la direction de **Sandrine PÉCHÉ**

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Résumé

Cette thèse est consacrée à l'étude de quelques modèles exactement solubles dans la classe d'universalité KPZ. Le premier chapitre dresse un panorama des méthodes récentes pour étudier ce type de systèmes. On présente aussi les différents travaux qui constituent cette thèse, sans rentrer dans les détails techniques, en insistant plutôt sur l'interprétation des résultats et les méthodes générales.

Ensuite viennent trois chapitres, correspondant à autant d'articles publiés ou soumis pour publication. Le premier chapitre est une étude asymptotique du système de particules en interaction q-TASEP, perturbé par des particules lentes. On montre que le système obéit au même type de théorème limite que le TASEP, et on observe une transition de phase appelée transition BBP. Le deuxième chapitre, basé sur des travaux en collaboration avec Ivan Corwin, introduit de nouveaux processus d'exclusion exactement solubles. Nous vérifions notamment les prédictions de la classe d'universalité KPZ, et nous nous intéressons aussi au comportement moins universel de la première particule. Le troisième chapitre correspond également à un travail en collaboration avec Ivan Corwin. Nous introduisons une marche aléatoire en environnement aléatoire, qui a la particularité d'être exactement soluble. Nous montrons que les corrections au second ordre au principe de grandes déviations vérifié par la marche sont distribuées selon la loi de Tracy-Widom. On donne une interprétation probabiliste de ce théorème limite, et on montre également que le résultat se propage à température nulle.

Abstract

This thesis is about exactly solvable models in the KPZ universality class. The first chapter provides an overview of the recent methods designed to study such systems. We also present the different works which constitute this thesis, leaving aside the technical details, but rather focusing on the interpretation of the results and the general methods that we use.

The three next chapters each correspond to an article published or submitted for publication. The first chapter is an asymptotic study of the q-TASEP interacting particle system, when the system is perturbed by a few slower particles. We show that the system obeys the same limit theorem as TASEP, and one observes the so-called BBP transition. The second chapter, based on a work in collaboration with Ivan Corwin, introduces new exactly solvable exclusion processes. We verify the predictions from KPZ scaling theory, and we also study the less universal behaviour of the first particle. The third chapter corresponds to a second work in collaboration with Ivan Corwin. We introduce a random walk in random environment, which turns out to be exactly solvable. We prove that the second order correction to the large deviation principle is Tracy-Widom distributed on a cube root scale. We give a probabilistic interpretation of this limit theorem, and show that the result also propagates at zero-temperature.

Remerciements

Je tiens tout d'abord à adresser mes plus vifs remerciements à ma directrice de thèse, Sandrine Péché, pour m' avoir proposé un sujet aussi motivant, avoir été toujours attentive à mon travail et constamment disponible pour m' aider, tout en me laissant une appréciable liberté.

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Contents

Avant-propos

La loi des grands nombres et le théorème central limite sont sans doute les deux théorèmes limites les plus importants de la théorie des probabilités. Ils décrivent tous deux le comportement asymptotique d'une somme d'aléas indépendants. Cependant, de nombreux phénomènes naturels sont le résultat d'aléas corrélés. Bien qu'il n'existe pas d'analogue du théorème central limite pour les variables aléatoires fortement corrélées, on peut délimiter certaines classes de modèles de systèmes aléatoires qui semblent obéir à des lois universelles. La classe d'universalité KPZ est l'une d'entre elles. Les premières investigations viennent de la littérature physique, en particulier les travaux de Kardar, Parisi et Zhang, en 1986, qui ont attiré beaucoup d'attention. La communauté mathématique s'est emparée de ces sujets à partir des années 2000, lorsque Johansson a découvert que les analogies entre la théorie des matrices aléatoires et la combinatoire des partitions d'entiers étaient fructueuses dans l'étude du TASEP. A priori, la classe d'universalité KPZ englobe principalement des modèles d'interfaces aléatoires. Cependant, en utilisant diverses bijections – notamment, de nombreux modèles de mécanique statistique sont canoniquement associées à une fonction de hauteur – elle contient de nombreux systèmes assez divers comme des polymères dirigés, des équations aux dérivées partielles stochastiques, des systèmes de particules en interaction ou encore certains modèles de percolation orientée. Une manière de relier ces modèles divers, hormis en mettant en évidence un modèle d'interface sous-jacent, est de montrer que les uns sont des limites des autres. Par exemple, certaines observables de systèmes de particules en interaction convergent vers la fonction de partition de modèles de polymères dirigés, et certains de modèles de percolation orientée sont des limites "en température nulle" de polymères dirigés.

Cette thèse est consacrée à l'étude mathématique de quelques modèles exactement solubles de systèmes de particules en interaction et de certaines de leurs limites. Tous ces modèles appartiennent à la classe d'universalité KPZ. Mon premier travail, présenté au Chapitre 2, est une analyse asymptotique du q-TASEP, un système de particules en interaction introduit par Borodin et Corwin en 2011. On s'intéresse au comportement d'un système perturbé par quelques particules plus lentes. Ce travail illustre les similitudes bien connues entre l'étude des valeurs propres de matrices aléatoires et des systèmes désordonnés.

L'étude des modèles exactement solubles de particules en interaction a connu des avancées spectaculaires ces dernières années, d'abord avec les travaux de Tracy et Widom à partir de 2008 sur le processus d'exclusion simple asymétrique (ASEP), puis avec l'introduction par Borodin et Corwin des processus de Macdonald. Dans un travail en collaboration avec Ivan Corwin présenté au Chapitre 3, nous introduisons une nouvelle classe de systèmes de particules en interaction sur le réseau $\mathbb{Z}$, que nous appelons le processus d'exclusion q-Hahn asymétrique. Il s'agit plus précisément, comme l'ASEP, d'un processus d'exclusion partiellement asymmétrique, c'est à dire concrètement que les particules peuvent sauter à droite et à gauche. Dans un cas particulier, nous retrouvons un modèle introduit par Sasamoto et Wadati, qui l'avaient étudié via l'ansatz de Bethe, et noté des similarités avec l'ASEP. Notre méthode nous permet de caractériser complètement la loi de la position des particules, et nous prouvons ainsi pour ce modèle les prédictions d'universalité concernant les fluctuations du système autour de sa limite hydrodynamique. Par ailleurs, Tracy et Widom avaient noté dès leurs travaux de 2008 que l'universalité semble être mise en défaut lorsqu'on s'intéresse à la première particule dans l'ASEP. Le processus q-Hahn asymétrique est un autre modèle exactement soluble de processus d'exclusion partiellement asymétrique. Ainsi, outre la vérification des prédictions de la classe d'universalité KPZ, notre travail permet de s'intéresser au comportement non-universel de la première particule.

Dans un deuxième travail en collaboration avec Ivan Corwin, nous introduisons une marche aléatoire en environnement aléatoire, où les probabilités de sauter de $+1$ ou -1 sont des variables aléatoires de loi Beta, indépendantes en chaque temps et en chaque site. Ce modèle est en quelque sorte une limite d'un système de particules en interaction introduit récemment, et il est exactement soluble. Nous montrons que les grandes déviations de la marche aléatoire doivent être corrigées au second ordre par un terme aléatoire. Ce terme aléatoire converge vers la loi de Tracy-Widom, dans l'échelle $t^{1/3}$ (t est le temps). Ce type de théorème limite est emblématique de la classe KPZ, ce qui laisse supposer que cette classe s'étend aux marches aléatoires en environnement aléatoire. Un autre aspect que nous étudions est la relation entre ces grandes déviations et le comportement du maximum de N points terminaux de marches aléatoires indépendantes, corrélées par un environnent commun. Ainsi, la loi de Tracy-Widom apparaît une fois de plus comme une loi limite pour les valeurs extrêmes de suites de variables aléatoires corrélées. Dans notre modèle, la structure de covariance est particulièrement simple, et quantitativement explicite. Nous présentons ces travaux au Chapitre 4, où nous étudions aussi la limite en température nulle de ce modèle de marche aléatoire, qui est un modèle de percolation dirigée de premier passage.

Foreword

The law of large numbers and the central limit theorem are certainly the two most important limit theorems in probability theory. Both describe the asymptotic behaviour of a sum of independent random variables. However, numerous natural phenomena result from correlated random variables. Although there does not exist an analogue of the central limit theorem for strongly correlated random variables, one can delineate certain classes of models of random systems which seem to obey universal laws. Among these classes, the KPZ universality class has been the subject of a lot of work. Early investigations into this class came from the physics community, in particular the work of Kardar, Parisi and Zhang in 1986, and interest has grown in the mathematics community from the 2000's. A priori, this class relates to random interface growth. However, via various mappings – many models from statistical mechanics are characterized by a height function for instance – it can be related to many systems such as directed polymers, stochastic partial differential equations, directed percolation, interacting particle systems, traffic models, etc. Apart from defining an underlying interface model, another way to connect the models in the KPZ universlity class is to show that the ones are limits of the others. For example, certain observables of interacting particle systems converge to the partition function of directed polymer models, and certain models of directed percolation are "zero-temperature limits" of directed polymers.

This thesis deals with the mathematical study of exactly solvable models of interacting particle systems, and some of their limits. All belong to the KPZ universality class. My first work, presented in Chapter 2, is an asymptotic analysis of the q-TASEP, an interacting particle system introduced by Borodin and Corwin in 2011. One is interested in the behaviour of a system perturbed by a few slower particles. This work illustrates the well-known close relationship and similarities between the study of eigenvalues of random matrices and disordered systems.

The study of exactly solvable interacting particle systems has made spectacular progress during the last years, first with the works of Tracy and Widom from 2008 on the asymmetric simple exclusion process (ASEP), and then with the introduction by Borodin and Corwin of Macdonald processes. In a joint work with Ivan Corwin presented in Chapter 3, we introduce a new class of interacting particle systems on the integer lattice, that we call the q-Hahn asymmetric exclusion process (q-Hahn AEP). It is more precisely, like ASEP, a partially asymmetric exclusion process, which means concretely that particles may jump to the right and to the left. In a particular case, we recover a model introduced by Sasamoto and Wadati, who had studied it via the Bethe ansatz, and noted similarities with ASEP. Our method allows to fully determine the law of particle's location, and we can thus prove the predictions from KPZ scaling theory for this model. Moreover, Tracy and Widom have remarked already in their 2008 works that universality seems to break down when one is interested in the asymptotic law of the first particle in the asymmetric simple exclusion process. The q-Hahn AEP is another exactly solvable partially asymmetric exclusion process. Thus, beyond the verification of KPZ universality predictions, our work allows to look at the non-universal behaviour of the first particle.

In a second work in collaboration with Ivan Corwin, we introduce a random walk in random environment on the integer lattice, where the probabilities of jumping by $+1$ or -1 are given by random variables following the Beta distribution, independently at each time and for each site. This model is in some sense a limit of an interaction particle system recently introduced, and it is exactly solvable. We show that the large deviations of the random walk endpoint are corrected at the second order by a random term. This term converge to the Tracy-Widom distribution, in the scale $t^{1/3}$ (t is the time). This type of limit theorem is emblematic of the KPZ universality class, suggesting that this class extends to random walks in random environment. Another aspect that we study is the relationship between large deviations and the behaviour of the maximum of a large number of random walk endpoints, correlated by a common environment. Thus, the Tracy-Widom distribution appears again as a limit for the extreme values of a sequence of correlated random variables. In our model, the covariance structure is particularly simple and quantitatively explicit. We present these works in the Chapter 4, where we also study the zero-temperature limit of the random walk, which is a directed first passage percolation model.

Chapter 1

Definitions and Main results

This chapter presents the main results contained in this thesis manuscript with twofold objectives:

1. We shall explain the motivations and place the results in their scientific context.
2. We try to focus on the intuitive meaning of the results and to identify the key arguments leading to their proofs.

In order to satisfy the first aim, the first section is devoted to explaining what the KPZ universality class is. In Section 1.2, we provide an introduction to exclusion processes, focusing on exactly solvable models. In Section 1.3, we move to another type of models, random directed lattice paths. In this section, we provide a very short introduction to random directed polymers and we explain how the exactly solvable models considered are related to interacting particle systems. The Section 1.4 is devoted to explaining the origins of exact solvability. We briefly describe an underlying algebraic structure which constitutes the theory of Macdonald processes, and explain the main methods used to reveal exact solvability. We try to unify the diverse models introduced by showing that they are characterized by very similar Fredholm determinantal formulas. In Section 1.5, we eventually present the limit theorems that we are able to prove, trying to focus on their probabilistic meaning.

1.1 KPZ universality

All stochastic models studied in this thesis belong to the so-called KPZ universality class. We provide here a short review about KPZ equation and KPZ universality class.

In 1986, Kardar Parisi and Zhang [KPZ86], studied the time evolution of random rough interfaces. They made scaling predictions, and claimed a form of universality. In one spatial dimension, the default model for such systems is a stochastic partial differential equation (the KPZ equation),

$$\partial_t h = \partial_{xx} h + (\partial_x h)^2 + \dot{\mathcal{W}}, \tag{1.1}$$

where $h(x,t)$ is supposed to be a function $\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ describing the height of the interface, and $\dot{\mathcal{W}}$ is a space time white noise. As initial condition, one can consider for instance the identically zero (flat) initial data. Without the non-linear term $(\partial_x h)^2$, Equation (1.1) would be a stochastic heat equation. In this case, one knows that the roughness of the noise propagates to the solution, which means that for a fixed time $t > 0$ the function $x \mapsto h(x,t)$ has Hölder exponent $1/2-\epsilon$. It is natural to expect the same regularity for the solution of the KPZ equation. Consequently, we lack a canonical way of making sense of the non-linear term $(\partial_x h)^2$. The challenge raised by the ill-posedness of KPZ equation has been recently solved by Martin Hairer [Hai13], and has led to the development of a theory of regularity structures [Hai14] which provides general tools to handle non-linearities in stochastic partial differential equations.

Although developing a rigorous framework for proving results such as existence and uniqueness is a very important task for the mathematician, another interesting problem is to describe quantitatively the solution. This is achieved by completely different techniques. It was observed that the logarithm of the solution of a multiplicative stochastic heat equation formally solves the KPZ equation. Thus Bertini and Giacomin [BG97] proposed to define the solution of the KPZ equation as the logarithm of a stochastic heat equation (SHE) with multiplicative noise. It happens that several discrete random systems converge (in a sense to be precised) to this type of SHE. Thus, a way of understanding the quantitative behaviour of the KPZ equation consists in studying related discrete systems.

Let us push this idea further. After all, the KPZ equation is not necessarily the most illuminating model for studying the growth of interfaces. For instance, the study of interacting particle systems such as the asymmetric simple exclusion process (ASEP), besides giving insight about the law of the KPZ equation [ACQ11, SS10], is interesting in itself: ASEP is also a default model for out-of-equilibrium transport phenomena.

In this thesis, we study discrete models that belong to the KPZ universality class. What does this mean ? One expects that a random process belongs to the KPZ universality class if it can be described by an interface (very often a height function), whose time evolution satisfies the following:

1. there is a smoothing mechanism, meaning that deep holes and sharp peaks tend to disappear, this corresponds to the Laplacian term $\partial_{xx}h$;
2. the growth is slope-dependent, such that the interfaces grows laterally, this corresponds to the term $(\partial_x h)^2$;
3. the randomness is driven by a noise with short range correlations in space and time.

All systems in the KPZ universality class exhibit a lot of common features. For instance, if the process is described by a height function $h(x,t)$, we expect that for a fixed point x, $h(x,t)$ has random fluctuations on the scale $t^{1/3}$ (opposed to the $\sqrt{t}$ scale for diffusions), and spatial decorrelation occurs on the $t^{2/3}$ scale. Moreover, the probability distributions and processes involved in limit theorems do not depend on the specificities of the model considered. It is delicate to make a very precise statement here, since limiting laws usually depend on initial conditions and symmetry properties, but it is worth mentioning that most limiting laws appeared previously in the study of fluctuations of extreme eigenvalues of random matrices. We refer to Corwin's review [Cor12] or Quastel's lecture notes [Qua] for a more complete discussion. For certain subclasses of the KPZ universality class, one is even able to make precise quantitative predictions, one case is treated in Section 1.5.1.

Let us conclude this introductory discussion of KPZ universality by saying that, outside a handful (perhaps with two hands in 2015) of exactly solvable models, universality claims

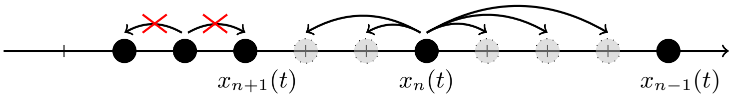

Figure 1.1: Description of a general exclusion process. Arrows represent allowed transitions for the nth particle.

are so far highly conjectural.

1.2 Integrable exclusion processes

This section provides an introduction to exclusion processes. We also introduce some general notations in order to define the exactly solvable exclusion processes studied in this thesis.

We consider interacting particle systems on $\mathbb{Z}$, described by coordinates of particles

$$(x_n(t))_{n\in\mathbb{Z}_+}.$$

The word exclusion means that each site can be occupied by at most one particle, and we assume that particles cannot overtake. Thus we can assume that

$$\cdots < x_n(t) < \cdots < x_1(t) < x_0(t) = +\infty.$$

We use for convenience a virtual particle indexed by 0 at $+\infty$. It implies in particular that there exists always a right-most particle. For the moment, we do not specify if jumps occur at discrete times or in continuous time, nor if we deal with a finite or infinite number of particles. In all examples considered, the dynamics are Markovian. When dealing with infinitely many particles, the definition of such Markov processes can be an issue, and we address this type of questions later (see Section 3.3). The simplest exclusion process is the totally asymmetric exclusion process (TASEP). This process was introduced initially in the study of protein synthesis [MGP68] to model the translation of ribosomes along a mRNA strand. Each particle possesses a clock which rings after an exponentially distributed random time of mean 1. When the clock rings, the particle jumps by one to the right, provided the target site is empty. If the site is already occupied by a particle, then the move is cancelled and one resets the clock.

The asymmetric simple exclusion process is an asymmetric variant. When the clock rings, the particle jumps to the right with probability q and to the left with probability p. Hence, each particle performs a simple random walk on $\mathbb{Z}$ with the caveat that all moves that would break the exclusion rule are cancelled. One can generalize the problem by assuming that the random walks may not be simple random walks but have transition kernel $p(x, y)$. The mathematical study of exclusion processes began with Spitzer [Spi70]. The books of Liggett [Lig06, Lig99] are the classical reference and present many properties of exclusion processes (well-definiteness, invariant measures, law of large numbers...).

In order to connect with the discussion from Section 1.1 about random interfaces, it is enlightening to explain how one can associate a growth model to an exclusion process. The mapping is initially due to Rost [Ros81]. We believe that Figure 1.2 is self-explanatory.

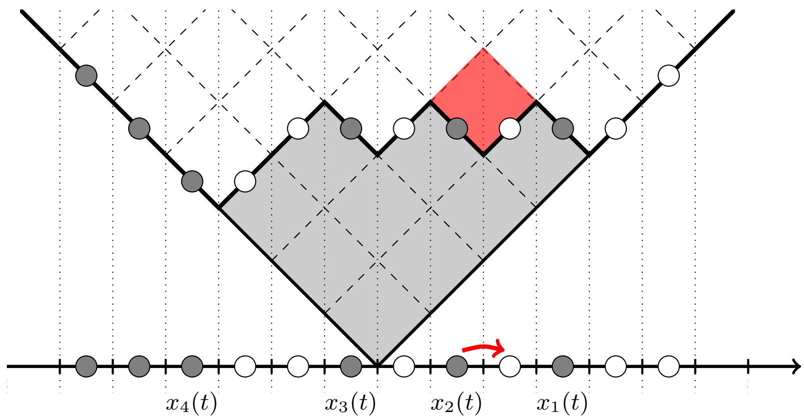

Figure 1.2: The border of the gray region constitutes the interface associated to the configuration of particles. One sees that a move of a particle by +1 to the right (resp. to the left) corresponds to adding (resp. removing) a box to the gray region.

1.2.1 General description

In the following, we describe exclusion processes in a slightly different way:

- **Discrete time totally-asymmetric case with parallel update:** If the nth particle sits in position $x_n(t)$ at time t, it can move to the site $x_n(t)+j$, for $j \in \{0, 1, \dots, x_{n-1}(t) - x_n(t) - 1\}$. We assume that all jumps occur independently and in parallel, which means that all particle locations are updated at the same time, and hence the position of a particle at time $t+1$ depends on the positions of its neighbour at time t but does not depend on their moves between times t and $t+1$. The process is characterized by the probabilities of these events which we denote as follows:

$$\varphi(j|m) := \mathbb{P}\Big(x_n(t+1) = x_n(t) + j \Big| x_{n-1}(t) - x_n(t) - 1 = m\Big).$$

  Note that there is not a canonical way to extend this definition in the partially asymmetric case.

- **Continuous time partially asymmetric case:** The process is described by the exponential rate of each possible transition. We denote by:
  - $\phi^R(j|m)$ the rate of the transition when the particle sitting in $x_n(t)$ jumps to the position $x_n(t)+j$, where $m = x_{n-1}(t) - x_n - 1$,
  - $\phi^L(j'|m')$ the rate of the transition when the particle sitting in $x_n(t)$ jumps to the position $x_n(t) - j'$, where $m' = x_n(t) - x_{n+1} - 1$.

The problem of studying the long-time behaviour of such systems depends on the initial condition that we consider. A priori, this initial condition can be random or deterministic, and involving a finite or infinite number of particles. In this manuscript, we restrict to a particular type of initial condition. We have already assumed that there exists a rightmost particle. More precisely, we focus on the step initial condition for which particles are initially packed on the negative integers:

$$\forall n \geqslant 1, x_n(0) = -n.$$

This initial condition is very far from equilibrium.

Several other initial conditions are interesting. One can start from a stationary measure, or start from a state which is not stationary but quickly relaxes to the stationary measure. Studying how the randomness of the initial condition propagates to the configuration of particles at time t is also interesting. In the following, we do not treat these questions. We are rather interested in understanding how the limit theorems depend on the microscopic dynamics, i.e. the particular choice of the probabilities (resp. rates) $\varphi(j|m)$ (resp. $\phi(j|m)$).

1.2.2 The q-TASEP

The q-TASEP was introduced by Borodin and Corwin in the context of Macdonald processes [BC14]. We refer to Section 1.4.1 for the connection to Macdonald processes. The q-TASEP is a continuous time Markov process described by the rates

$$\phi^R(j|m) = \big(1-q^m\big)\mathbb{1}_{\{j=1\}} \quad \text{and} \quad \phi^L(j|m) = 0.$$

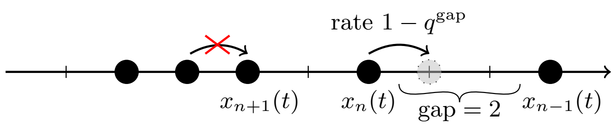

Figure 1.3: Illustration of a possible move in the q-TASEP. The gap in front of the nth particle is the quantity $x_{n-1}(t)-x_n(t)-1$, that is the number of consecutive empty sites up to the next particle on the right.

The q-TASEP is exactly solvable, as we will see in Section 1.4.1, even if one allows the particles to have different speeds. More precisely, one can assume that the nth particle jumps by one to the right with rate $a_n(1-q^{gap})$. This observation is very important in the chapter 2, where we study a q-TASEP with a finite number of slow particles (i.e. for all but finitely many n, the speed a_n is smaller than 1, and $a_n = 1$ for all other particles). When q equals zero, one recovers the TASEP. The parameter q can be interpreted as a repulsion strength between particles. When q goes to 1, jump rates go to zero and the process does not seem very interesting. However, if one simultaneously accelerates time by a factor $(1-q)^2$, one recovers after an appropriate renormalization (see [BC14, Chapter 5]) the O'Connell-Yor semi-discrete directed polymer introduced in [OY01]. Further scaling limits lead to the continuum directed polymer, whose free energy solves the KPZ equation.

1.2.3 Introduction to q-analogues

This Section contains a superficial introduction to the theory of q-analogues. It provides definitions that will be used in order to define some integrable interacting particle systems and the q-Hahn probability distribution. Moreover, the parallelism between the q-deformed world and the classical world provides much intuition when it comes to taking limits of (q-deformed) interacting particle systems in Section 1.3.

A q-analogue of a mathematical object is a generalization of this object depending on a parameter q – a priori a complex number – such that one recovers the classical object when

q tends to 1. It is preferable that the generalization preserve the main properties of the initial object. Moreover, q-analogues should ideally be compatible between themselves. For instance the q-derivative of the q-exponential function should be the q-exponential function. Very often, q-analogues are discretizations of their corresponding classical object, which explains their potential value in the study of random discrete processes.

Let us start with the q-analogues of elementary combinatorics. Given an integer n, the q-integer $[n]_q$ is

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

Quite naturally, one defines the q-factorial as

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

Since we will make an extensive use of q-factorials, it is convenient to define the q-Pochhammer symbol

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \qquad \text{and} \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

so that

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}.$$

By analogy with the classical binomial coefficients, the q-binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Let us mention a first interesting result:

Lemma 1.2.1 ([Sch53]). *Consider an associative algebra over the complex numbers*[1] *generated by two elements X and Y such that $YX = qXY$. Then one can always develop the product $(X+Y)^n$ as a sum of monomials of the form X^kY^{n-k}. One has the q-binomial expansion*

$$(X+Y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q X^kY^{n-k}.$$

A generalization of Lemma 1.2.1, presented in the next Section, is used several times in this thesis in order to diagonalize Markov generators via Bethe ansatz.

Let us define a few other q-deformations. For convenience, fix hence forth that $q \in (0,1)$. We define the q-derivation of a function $\mathbb{C} \to \mathbb{C}$ by the formula

$$\mathrm{d}_q f(x) = \frac{f(qx) - f(x)}{qx - x}.$$

1. The scalar field has no importance in the lemma.

In this way, the q-derivative of the monomial X^n is, as we should expect, $[n]_q X^{n-1}$. It follows that if we define the q-exponential function as

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!},$$

then $\mathrm{d}_q e_q(x) = e_q(x)$ as desired. Note that the above series is convergent when $|x| < 1/(1-q)$. For x in a compact set, the q-exponential function converges uniformly to the exponential as q goes to 1. For a complex number z with $|z| < 1$, the q-binomial theorem [AAR99, Theorem 10.2.1] implies that

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.2}$$

Given the formula (1.2) we have the product representation

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty}.$$

For a random variable X, the e_q-Laplace transform of X is defined in [Hah49] as

$$\mathbb{E}\left[e_q(\zeta X)\right] = \mathbb{E}\left[\frac{1}{((1-q)\zeta X;q)_\infty}\right].$$

Under certain conditions [2], the q-Laplace transform of a positive random variable X is analytic over $\mathbb{C}\setminus\mathbb{R}_+$. This property proves very useful for converting combinatorial properties encapsulated in formal generating series, into formulas amenable to asymptotic analysis. An inversion formula of the e_q-Laplace transform is given in [BC14].

1.2.4 The q-Hahn TASEP and the q-Hahn distribution

The q-Hahn TASEP is introduced (although under another name) by Povolotsky in [Pov13] and is further studied by Corwin in [Cor14]. This is a discrete-time exclusion process with parallel update, where particles jump only to the right. Before giving the precise expression for the transition probabilities $\varphi^R(j|m)$, let us explain the motivations. In a short letter [EMZ04], Evans, Majumdar and Zia consider spatially homogeneous discrete time zero-range processes on periodic domains. They address and solve the question of characterizing the jump distributions for which invariant measures are product measures. This is important because the existence of invariant product measure is often the first requirement for exact-solvability. Povolotsky [Pov13] further examined the most general form of jump distributions allowing solvability by Bethe ansatz, and finds a family depending on three real parameters q, μ and ν, that we introduce now.

Definition 1.2.2. Let q, μ, ν be three parameters such that $q \in (0,1)$ and $0 \leqslant \nu \leqslant \mu < 1$. Let m be some positive integer. The q-Hahn distribution is a probability distribution on the set $\{0, 1, \dots, m\}$ where the probability of drawing j is given by

$$\varphi_{q,\mu,\nu}(j|m) = \mu^j \frac{(\nu/\mu;q)_j (\mu;q)_{m-j}}{(\nu;q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q.$$

2. The conditions are quite strong. In the cases encountered later, X is of the form $X = q^Z$ where Z is a random variable on positive integers, so that X is bounded.

The weights $\varphi_{q,\mu,\nu}(j|m)$ are introduced in the context of integrable particle systems by Povolotsky [Pov13] as an analogue of binomial coefficients[3] in an algebra satisfying the quadratic homogeneous relation

$$YX = \alpha XX + \beta XY + \gamma YY.$$

More precisely, it is shown in [Pov13, Theorem 1] that if

$$\alpha = \frac{\nu(1-q)}{1-q\nu}, \quad \beta = \frac{q-\nu}{1-q\nu}, \quad \gamma = \frac{1-q}{1-q\nu},$$

and

$$\mu = p + \nu(1-p),$$

then

$$\big(pX + (1-p)Y\big)^n = \sum_{k=0}^{n} \varphi_{q,\mu,\nu}(j|n) X^k Y^{n-k}.$$

This binomial formula is very useful in Chapters 3 and 4. It allows to factor the action of Markov transition kernels or generators (see more precisely Proposition 3.3.9 in Section 3.3, and Section 4.3.1).

In light of the connection between the q-Hahn distribution and a q-analogue of the Pólya urn scheme (See [GO09, Section 4] and Section 3.2.2), the q-Hahn distribution should be seen as a q-analogue of the Beta-Binomial distribution. Moreover, one could express the weights $\varphi_{q,\mu,\nu}(j|m)$ in terms of q-Gamma functions (the canonical q-analogue of the Gamma function), making the connection to the Beta binomial even more explicit. This distribution enjoys a particular symmetry that we discuss in Section 3.2.2 (we also give basic properties such as a formula for the expectation of the q-Hahn distribution).

Digression 1.2.3. *Let us briefly address the reason behind the name q-Hahn. The q-Hahn orthogonal polynomials* $(Q_n(x))_n$ *are orthogonal polynomials in the variable* q^{-x} *and they are defined by*

$$Q_n(x) \equiv Q_n(x; a, b, N; q) = {}_3\phi_2 \left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x} \\ aq, q^{-N} \end{matrix} ; q, q \right],$$

where ${}_3\phi_2$ *is a basic hypergeometric series (see [GR04]). They satisfy an orthogonality relation [GR04, (7.2.22)] that we phrase in a probabilistic way. Set parameters* $\mu = bq$, $\nu = abq^2$ *and let* X *a random variable following the q-Hahn distribution on* $\{0, 1, \dots, m\}$. *Then*

$$\mathbb{E}\left[Q_n(X) Q_{n'}(X)\right] = \mathbb{1}_{n=n'} \|Q_n\|^2,$$

where we tautologically define the norm of q-Hahn polynomials by

$$\|Q_n\|^2 = \mathbb{E}\left[\big(Q_n(X)\big)^2\right].$$

Thus, q-Hahn orthogonal polynomials are orthogonal with respect to the q-Hahn distribution[4].

3. The binomial formula was also derived in [Ros00].

4. The notational conventions that we use for the weight function and $\|Q_n\|$ slightly differs from [BCPS14] and from [VK93].

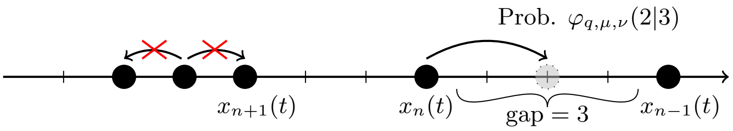

Figure 1.4: Illustration of the q-Hahn TASEP.

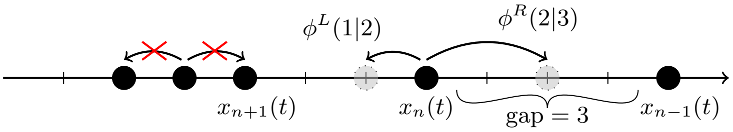

Figure 1.5: Illustration of the asymmetric q-Hahn exclusion process.

It should not be surprising now that we define the transition probabilities in the totally asymmetric q-Hahn exclusion process by

$$\varphi(j|m) = \varphi_{q,\mu,\nu}(j|m).$$

1.2.5 The asymmetric q-Hahn exclusion process

The (continuous-time) asymmetric q-Hahn exclusion process (abbreviated q-Hahn AEP) is introduced in [BC15a], in a work in collaboration with Ivan Corwin. The aim of this work was to generalize the construction and the solvability of the q-Hahn TASEP to processes where particles can jump both to the right and to the left. In the discrete-time setting, the exact solvability does not perfectly extend to the partially asymmetric case. But we can define a family of exactly solvable continuous time exclusion processes determined by the rates

$$\begin{aligned}
\phi^R(j|m) &:= R\,\frac{\nu^{j-1}}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}},\\
\phi^L(j|m) &:= L\,\frac{1}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}}.
\end{aligned}$$

These rates are constructed as limits of q-Hahn probabilities as we explain in Chapter 3. Setting $\nu = 0$ and asymmetry parameters $L = 0$ and $R = 1$, the rates of jumps have the simple form

$$\phi^R\big(j|m\big) = (1-q^m)\mathbb{1}_{\{j=1\}},\quad \phi^L\big(j'|m'\big) = 0$$

matching those of q-TASEP. However, when $L > 0$, jumps on the left are long-range. Hence our two-sided dynamics are different from those of the classical asymmetric simple exclusion process, but rather generalize the PushASEP [BF08].

1.2.6 Multi-particle asymmetric diffusion model

Setting $\nu = q$ in the rates of the asymmetric q-Hahn exclusion process, they no longer depend on the distance to the neighbouring particles and have a particularly simple form.

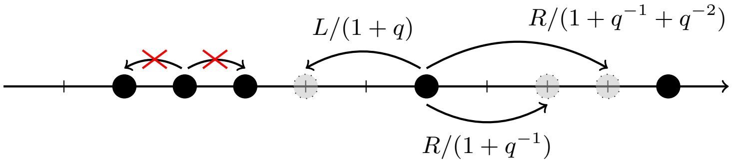

Figure 1.6: Rates of a few admissible jumps in the exclusion process corresponding to the multi-particle asymmetric diffusion model (MADM exclusion process).

The MADM exclusion process is defined by the rates

$$\phi^R(j|m) := \frac{R}{[j]_{q^{-1}}},$$
$$\phi^L(j|m) := \frac{L}{[j]_q}.$$

An example of some possible jumps is shown in Figure 1.6. One of our motivations for studying this model is that it has been known to be exactly solvable for a long time. Indeed, Sasamoto and Wadati [SW98b] introduced a one-parametric family of zero-range processes diagonalizable via Bethe ansatz, called the multi-particle asymmetric diffusion model (MADM). Using a classical coupling between zero-range and exclusion processes (see Section 1.4.2) that maps the gaps between consecutive particles $x_i - x_{i+1} - 1$ in the exclusion process with the population of the i^{th} site in the zero-range process, the MADM corresponds to the q-Hahn AEP with $R = q/(1+q)$ and $L = 1/(1+q)$ (and $\nu = q$). It was later extended to arbitrary asymmetry parameters $R, L > 0$ [AKK99], and further studied in [Lee12]. Until [BC15a], no formulas amenable to asymptotic analysis had been written down for these systems.

1.3 Directed lattice paths

In this section, we introduce a few models of directed random lattice paths: random walks in random environment and directed random polymers. Although the subject seems different, we explain that the models that we consider are actually limits of interacting particle systems, and can be studied in a parallel way.

Directed random polymers have been introduced in [HH85] as a model for the interface of the two-dimensional Ising model with random interactions. In general, we fix a lattice, say $\mathbb{Z}_+ \times \mathbb{Z}^d$ and we consider paths between the point $(0,0)$ and a point (t,x) for some $x \in \mathbb{Z}^d$. Let P_t be the law of the nearest-neighbour random walk on $\mathbb{Z}^d$ up to time t. It defines naturally a measure on paths of $\mathbb{Z}_+ \times \mathbb{Z}^d$ between $(0,0)$ and $\{t\} \times \mathbb{Z}^d$. For any such path π, we associate an energy

$$H_t(\pi) = \sum_{e \in \pi} w_e,$$

where the sum runs over all edges in π, and the weights w_e are random variables that constitute the environment. A directed polymer is then a measure Q_t on paths given by

$$\frac{\mathrm{dQ}_t}{\mathrm{dP}_t}(\pi) = \frac{\exp(-\beta H_t(\pi))}{Z_t},$$

where the normalisation constant Z_t is called the partition function, and β is called the inverse temperature. Usually, $F_t := \log(Z_t)$ is called the free energy, and many properties of the dependency of the polymer paths on its environment are encoded in the analytical properties of the function

$$f(\beta) = \lim_{t\to\infty} \frac{F_t}{t}.$$

The literature on random directed polymers is vast, and our definition is restrictive: there exist models for which the paths does not necessarily jump to one of the nearest-neighbours, the weights are not necessarily attached to the edges but sometimes to the vertices [5], and polymer paths do not even necessarily live on a lattice! Note also that sometimes the parameter β is not present, but the temperature is implicitly defined in terms of the parameters of the random variables w_e.

A nearest-neighbour directed random walk in random environment on $\mathbb{Z}_+ \times \mathbb{Z}^d$, is a random walk which performs steps of the form [6] $(1, y)$ for some $y \in \mathbb{Z}^d$ with $\|y\| = 1$. For each edge e of the form $(t, x) \to (t + 1, x + y)$ (where $\|y\| = 1$), we assign a random transition probability p_e. More precisely, we consider a random walk $(X_t)_t$ such that if $X_t = x$, then $X_{t+1} = X_t + e$ with probability p_e. The random walk is well-defined provided that from every point (t, x) the outgoing probabilities sum to 1, i.e.

$$\sum p_e = 1$$

where the sum runs over all edges of the form $(t, x) \to (t + 1, x + y)$. It is interesting to notice that if we consider a directed polymer with $\beta = 1$ and weights w_e which can be written

$$w_e = -\log(p_e),$$

then the polymer is a directed RWRE.

For general directed polymer models, it is customary to define the annealed free energy by

$$\lambda(\beta) = \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[Z_t]),$$

where the expectation is taken over the environment. When the weights w_e are i.i.d., $\lambda(\beta)$ has a very simple expression. The weak disorder regime consists of β such that

$$\lim_{t\to\infty} e^{-t\lambda(\beta)} Z_t > 0,$$

whereas the strong disorder regime is when the limit is zero. The strong disorder regime can also be seen as a strict ordering between quenched and annealed free energies [CSY04]

$$f(\beta) < \lambda(\beta).$$

One can think of $f(\beta)$ as the analogue of the large deviation rate function. Actually, it is precisely a LDP rate function when the polymer is a directed RWRE. The randomness of the environment appears in the free energy at the second order, and one conjectures that

$$F_t = tf(\beta) + t^{\chi} X_t,$$

5. This is actually more common, but can be seen as a particular case of the situation where weights are attached to edges.

6. the first coordinate, in $\mathbb{Z}_+$, is the direction of the walk. It will later be interpreted as the time direction for random walks in space-time random environment.

where X_t weakly converges to a non-degenerate random variable, and χ is called the *longitudinal fluctuation exponent*. In all examples that we discuss below, X_t converges to the Tracy-Widom distribution (see Section 1.5). Another interesting – and related – aspect of directed polymers is their localization properties. Conditionally on the environment, the polymer path is (in strong disorder) subdiffusive, i.e. it concentrates in a region of size $o(\sqrt{t})$, but when averaging over the environment, one sees that the polymer will search for rewards in the energy landscape at a much larger scale. In order to formalize this, one assumes that their exists a *transversal fluctuation exponent* ζ such that when we let the environment vary, the endpoint of the polymer varies in a region of size t^{ζ}. For polymers in $1+1$ dimension, one expects under assumptions on the environment that $\chi = 1/3$, $\zeta = 2/3$, so that the fluctuation exponents should verify the relation

$$\chi = 2\zeta - 1.$$

This relation is called the KPZ relation, and it is conjectured to hold in any dimension (So far, the only progress towards a proof of ths conjecture in a general setting deals only with first passage percolation [Cha13] – which is a sort of zero-temperature limit – and relies on strong assumptions).

Since we are unable to prove such results at positive temperature for general polymers, it is interesting to better understand a few exactly solvable models for which we can find the scaling exponents and much more.

The first example of an exactly-solvable polymer model is the semi-discrete O'Connell-Yor polymer [OY01]. It corresponds to a limit of polymer in $\mathbb{Z}_+^2$ where one would have rescaled diffusively only one of the coordinates. A connection between this model and Whittaker functions was pioneered in [OC12]. As Whittaker functions are a degeneration of Macdonald functions, it was shown using general techniques developed for Macdonald processes (see Section 1.4.1) that the longitudinal fluctuation exponent is $1/3$ and the law of the rescaled free energy converges to the Tracy-Widom distribution [BC14, BCF14]. It is important for our purpose to mention that this model is related to interacting particle systems, in the sense that some observables of the q-TASEP converge, after an appropriate scaling, to the partition function of the semi-discrete polymer. There exists also a continuous limit of the semi-discrete polymer. In this model, polymer paths are Brownian paths, and the weight of a path is the integral of a white noise along the path. Via a Feynman-Kac representation, one can show that the partition function for this model solves a heat equation with multiplicative noise, and hence the free energy solves the KPZ equation [7].

The first discovered exactly-solvable polymer model on a lattice is the log-gamma polymer, introduced in [Sep12]. This is a discrete polymer model in which the weights are attached to vertices and distributed as the logarithms of inverses of i.i.d. Gamma random variables (so that the probability of a given path is proportional to a product of Gamma weights along the path). For this model, the KPZ relation is satisfied [Sep12], and one can prove as-well a Tracy-Widom limit theorem [COSZ14, BCR13].

A second exactly solvable model, which is related to the log-gamma polymer, is the strict-weak lattice polymer, studied independently in [OO15] and [CSS15]. In [CSS15],

7. The polymer interpretation of the solution to the heat equation with multiplicative noise is explained in [KS91, Section 5.2]. A rigorous mathematical analysis of the SHE with multiplicative noise, in relation to KPZ equation, is done in [BC95], while a mathematical construction of the directed polymer model is done in [AKQ14].

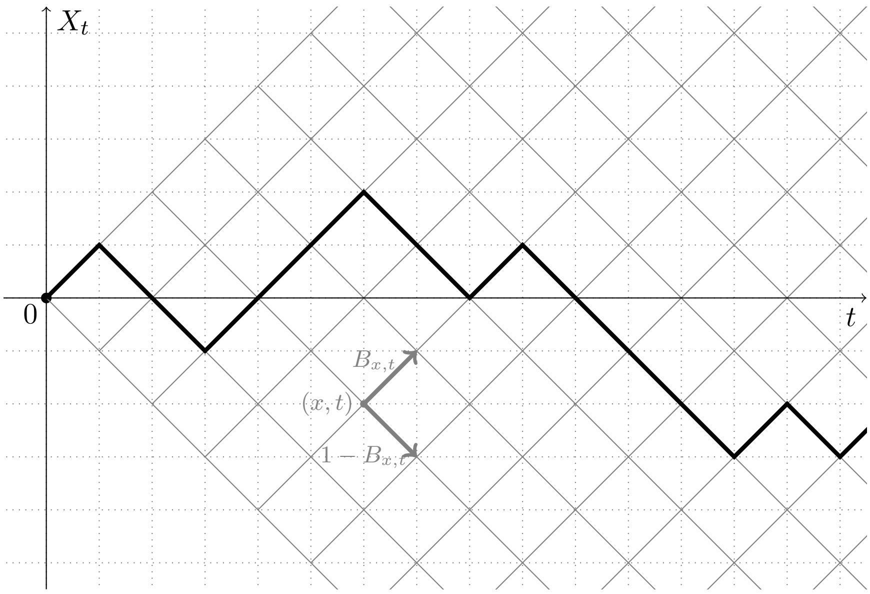

Figure 1.7: The graph of $t \mapsto X_t$ for the Beta RWRE. It is evident on the figure that $\mathbf{X}_t := (t, X_t)$ defined a directed random walk in a random environment in $\mathbb{Z}^2$.

it is shown that some observables of the discrete-time geometric q-TASEP converge to the partition function of the polymer, and hence one obtains formulas for the Laplace transform of Z_t by taking limits of geometric q-TASEP Fredholm determinant formulas.

Since we know that there exists a generalization of the geometric q-TASEP, that is the q-Hahn TASEP, it is natural to ask if one can define a new exactly solvable polymer model by taking limits of q-Hahn TASEP observables. The answer is positive, and we describe in the next sections the corresponding model and some limits, introduced in [BC15b].

1.3.1 Random walk in space-time i.i.d. Beta environment

Directed random walks in random environment can also be seen as random walks in space-time random environment. Let $(B_{x,t})_{x\in\mathbb{Z},t\in\mathbb{Z}_+}$ be a collection of independent random variables following the Beta distribution, with parameters α and β, i.e.

$$\mathbb{P}\left(B_{0,0} \leqslant r\right) = \int_0^r x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\mathrm{d}x.$$

We call this collection of random variables the environment of the walk. In this environment, we define the random walk in space-time Beta environment (abbreviated Beta-RWRE) as a random walk $\big(X_t\big)_{t\in\mathbb{Z}_+}$ in $\mathbb{Z}$, starting from 0 and such that

- $X_{t+1} = X_t + 1$ with probability $B_{X_t,t}$ and
- $X_{t+1} = X_t - 1$ with probability $1 - B_{X_t,t}$.

We will denote P and E (resp. $\mathbb{P}$ and $\mathbb{E}$) the measure and expectation related to the random walk (resp. to the environment). The random walk $(\mathbf{X}_t)_t$ in $\mathbb{Z}^2$, where $\mathbf{X}_t := (t, X_t)$ is a directed random walk in random environment (see Figure 1.7).

The Beta-RWRE is closely related to the q-Hahn TASEP. Indeed, let $\big(x_n(t)\big)$ be the particles location in the q-Hahn TASEP with parameters $(q, \bar{\mu}, \bar{\nu})$. We now fix two positive real parameters $\alpha, \beta > 0$ and assume that

$$\bar{\mu} = q^{\alpha}, \quad \bar{\nu} = q^{\alpha+\beta}.$$

Then, for any $t \in \mathbb{Z}_{\geqslant 0}$ and $n \in \mathbb{Z}_{>0}$, we have the weak convergence as q goes to 1,

$$q^{x_n(t)} \Longrightarrow \mathsf{P}(X_t \geqslant t - 2n + 2).$$

where $\mathsf{P}(X_t > t - 2n + 2)$ is the probability that $X_t \geqslant t - 2n + 2$. It is itself a random variable depending on the environment.

1.3.2 Beta polymer

We now describe a quite particular directed polymer model in $(Z_+)^2$, which is very useful in the study of the Beta-RWRE. This is a generalisation of the strict-weak lattice polymer described in [OO15, CSS15].

A point-to-point Beta polymer is a measure $Q_{t,n}$ on lattice paths π between $(0,0)$ and (t,n). At each site (s,k) the path is allowed to

- jump horizontally to the right from (s,k) to $(s+1,k)$,
- or jump diagonally to the upright from (s,k) to $(s+1,k+1)$.

An admissible path is shown in Figure 1.8. Let $B_{i,j}$ be independent random variables distributed according to the Beta distribution with parameters α and β where $\alpha, \beta > 0$. The measure $Q_{t,n}$ is defined by

$$Q_{t,n}(\pi) = \frac{\prod_{e\in\pi} w_e}{Z(t,n)}$$

where the weights w_e are defined by

$$w_e = \begin{cases} B_{ij} & \text{if } e = (i-1,j) \to (i,j) \text{ (horizontal edge)} \\ 1 & \text{if } e = (i-1,i) \to (i,i+1) \text{ (boundary edge)} \\ 1 - B_{i,j} & \text{if } e = (i-1,j-1) \to (i,j) \text{ with } i \geqslant j \text{ (up-right edge)}, \end{cases}$$

and $Z(t,n)$ is a normalisation constant called the partition function,

$$Z(t,n) = \sum_{\pi:(0,1)\to(t,n)} \prod w_e.$$

The Beta polymer is also related to the q-Hahn TASEP, even more closely than the Beta-RWRE. Consider $\big(x_n(t)\big)$ to be the particles locations in the q-Hahn TASEP with parameters $(q, \bar{\mu}, \bar{\nu})$. Assume that

$$\bar{\mu} = q^{\mu}, \quad \bar{\nu} = q^{\nu}.$$

Then, for any fixed parameters $\nu > \mu > 0$, we have the weak convergence of processes,

$$\left(q^{x_n(t)}\right)_{n>0,t\geqslant 0} \Longrightarrow \Big(Z(t,n)\Big)_{n>0,t\geqslant 0}, \text{ when } q \to 1.$$

Although the Beta polymer model was discovered through a limit of the q-Hahn TASEP, it can be analyzed independently. All the steps leading to the exact solvability of the q-Hahn TASEP can be perfomed in the $q \to 1$ limit. This was not the case for the limit of the geometric q-TASEP. This is because in the case of the semi-discrete, log-gamma and strict-weak directed polymers, the moments of the partition function grow too fast to determine uniquely the distribution.

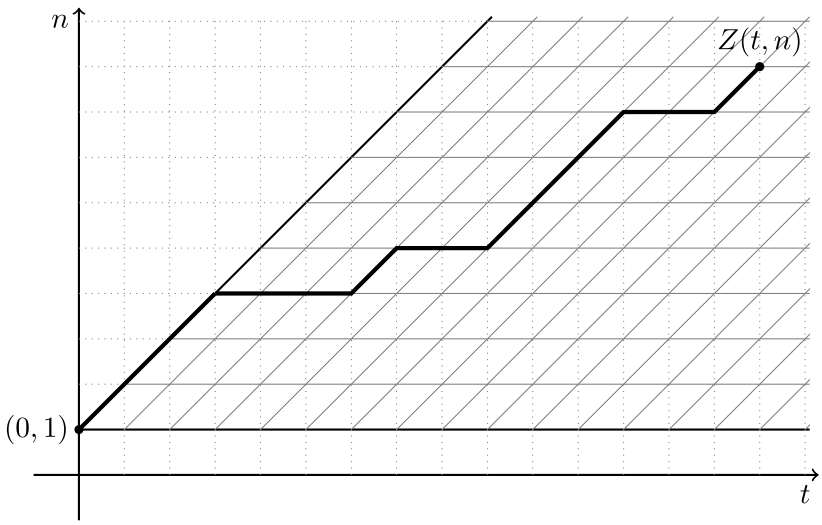

Figure 1.8: The thick line represents a possible polymer path in the point-to-point Beta polymer model.

Remark 1.3.1. The strict-weak lattice polymer is the limit (modulo a scaling of the partition function by a power of ν) of the Beta polymer, see Remark 4.1.5.

1.3.3 Bernoulli-Exponential directed first passage percolation

Let us introduce the "zero-temperature" counterpart of the Beta RWRE. Let (E_e) be a family of independent exponential random variables indexed by the horizontal and vertical edges e in the lattice $\mathbb{Z}^2$, such that E_e is distributed according to the exponential law of parameter a if e is a vertical edge and E_e is distributed according to the Exponential law of parameter b if e is a horizontal edge. Let $(\xi_{i,j})$ be a family of independent Bernoulli random variables with parameter $b/(a+b)$. For an edge e of the lattice $\mathbb{Z}^2$, we define the passage time t_e by

$$t_e = \begin{cases} \xi_{i,j} E_e \text{ if } e \text{ is the vertical edge } (i,j) \to (i,j+1), \\ (1-\xi_{i,j}) E_e \text{ if } e \text{ is the horizontal edge } (i,j) \to (i+1,j). \end{cases} \tag{1.3}$$

The first passage-time $T(n,m)$ in the Bernoulli-Exponential first passage percolation model is given by

$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e,$$

where the minimum is taken over all up/right path π from $(0,0)$ to $D_{n,m}$, which is the set of points

$$D_{n,m} = \Big\{ (i, n+m-i) : 0 \leqslant i \leqslant n \Big\}.$$

The percolation cluster $C(t)$ is defined by

$$C(t) = \Big\{ (n,m) : T(n,m) \leqslant t \Big\}.$$

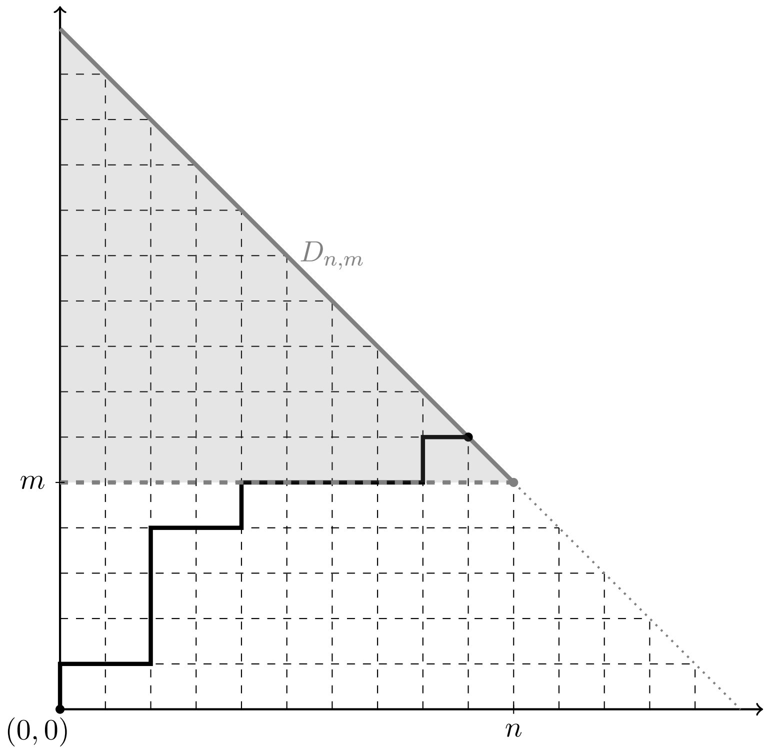

Figure 1.9: An admissible path for the Bernoulli-Exponential FPP model is shown on the figure. $T(n,m)$ is the passage time between $(0,0)$ and $D_{n,m}$ (thick gray line).

It can be constructed in a dynamic way (see Section 1.3.3). At each time t, $C(t)$ is the union of points visited by (portions of) several directed up/right random walks in the quarter plane $\mathbb{Z}_+^2$.

Remark 1.3.2. When b tends to infinity, E_e tends to 0 for all vertical edges, and one recovers the first passage percolation model introduced in [OCo99], which is the zero temperature limit of the strict-weak lattice polymer as explained in [OO15, CSS15].

Why is the Bernoulli-Exponential first passage percolation model considered as the zero-temperature limit of the Beta-RWRE ? Let us set $\alpha_\epsilon = \epsilon a$ and $\beta_\epsilon = \epsilon b$. The parameter ϵ plays the role of the temperature. Let X_t a Beta-RWRE with parameters α_ϵ and β_ϵ, and $T(n,m)$ the first-passage time in the Bernoulli-Exponential FPP model with parameters a, b. Then, for all $n, m \geqslant 0$, we have the weak convergence

$$-\epsilon \log \Big(\mathsf{P}(X_{n+m} \geqslant m-n) \Big) \Longrightarrow T(n,m),$$

as ϵ goes to zero, where $\mathsf{P}(X_{n+m} \geqslant m-n)$ is the probability for the Beta-RWRE to be above the site $m-n$ at time $n+m$. Recall that $\mathsf{P}(X_{n+m} \geqslant m-n)$ is random since it depends on the environment.

1.4 Exact solvability: origins and methods

In order to step back on the definitions given in the previous sections, it is essential to introduce the reader to the theory of Macdonald processes. However, since we do not use this theory so much, we explain in Section 1.4.2 another approach leading to similar results. We call this approach the duality method. We then explain the outcome of these methods: Fredholm determinantal formulas. Most of the concepts and methods that we present in this section are introduced in [BC14, BCS14], but have a wider range of applicability than Macdonald processes and exclusion processes.

1.4.1 Macdonald processes

In order to connect the study of q-TASEP with its algebraic origin, we give here an impressionistic introduction to Macdonald processes. The reader is referred to the seminal paper [BC14], which introduces Macdonald processes and explain several applications. Two surveys on Macdonald processes and so called “integrable probability” are also available: [BG12] focuses slightly more on applications to members of the KPZ universality class, whereas [BP14] make the inspirations from representation theory more explicit.

An interlacing triangular array is a sequence of integer partitions $\big(\lambda^{(i)}\big)_{i=1,\dots,N}$ where each $\lambda^{(i)}$ is an integer partition

$$\lambda^{(i)} = \big(\lambda_1^{(i)} \geqslant \lambda_2^{(i)} \geqslant \dots \geqslant \lambda_i^{(i)} \geqslant 0\big).$$

The sequence has to satisfy the interlacing condition:

$$\begin{array}{ccccccccccc}
\lambda^{(N)}_N & & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & & \lambda^{(N)}_1 \\
& \geqslant \quad \leqslant & & \geqslant \quad \leqslant & & & \geqslant \quad \leqslant & & & \geqslant \quad \leqslant & \\
& & \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & \cdots & \lambda^{(N-1)}_2 & & \lambda^{(N-1)}_1 & & \\
& & & \geqslant \quad \leqslant & & & & \geqslant \quad \leqslant & & & \\
& & & & \ddots & & \iddots & & & & \\
& & & & \geqslant \quad \leqslant & & \geqslant \quad \leqslant & & & & \\
& & & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & & \\
& & & & & \geqslant \quad \leqslant & & & & & \\
& & & & & \lambda^{(1)}_1 & & & & &
\end{array}$$

Macdonald processes are a family of measures supported on such interlacing triangular arrays. The probability of a given triangular array is expressed in terms of Macdonald symmetric functions, which have been introduced in [Mac95], and are usually denoted P_λ and Q_λ. The Macdonald functions are a family of symmetric functions in countably many variables, indexed by (skew) integer partitions, such that the coefficient of each monomial is a rational fraction in $\mathbb{Q}(q,t)$ where $q,t \in (0,1)$ are two parameters. When $q = t$, Macdonald functions degenerate to Schur functions and the construction of Borodin and Corwin specializes to the Schur process introduced in [Oko01, OR03].

Macdonald functions satisfy a Cauchy identity: for two sequences of independent variables $x_1, x_2, \dots$ and $y_1, y_2, \dots$,

$$\sum_\lambda P_\lambda(x) Q_\lambda(y) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},$$

where the sum runs over all integer partitions. If one can choose the variables $x_1, x_2, \dots$ and $y_1, y_2, \dots$ so that each term $P_\lambda(x)Q_\lambda(y)$ is non-negative, then the Cauchy identity yields a very natural way of constructing a measure on integer partitions. It turns out that one knows a very large family [8] of sequences of variables for which $P_\lambda(x)Q_\lambda(y) \geqslant 0$ for any λ. Since we are dealing with possibly infinitely many variables, it is not convenient to keep these variables x and y. However, since the Macdonald functions form a basis of the space of symmetric functions, choosing $x_1, x_2, \dots$ corresponds to choosing a homomorphism from the space of symmetric functions to $\mathbb{C}$. This morphism is called a specialization. It turns out that a large family of specializations sending the Macdonald functions to the non-negative reals are described by a tuple (α, β, γ), where $\alpha = (\alpha_i)_{i \geqslant 1}$ and $\beta = (\beta_i)_{i \geqslant 1}$ are sequences of non-negative reals such that $\sum \alpha_i + \beta_i < \infty$ and γ is a non-negative real.

Macdonald processes are constructed so that the probability that the nth level is an integer partition λ is proportional to $P_\lambda(a_1, \dots, a_n)Q_\lambda(\rho)$, where $a_1, \dots, a_n$ are non-negative parameters and $Q_\lambda(\rho)$ corresponds to applying the specialization ρ to Q_λ, and ρ is described as above by a tuple (α, β, γ). The probability of the entire triangular array is given in terms of products of Macdonald functions, such that the process of $n \mapsto \lambda^{(n)}$ is Markov in the space of integer partitions. Macdonald processes have the following very nice property: Consider the measures on triangular arrays described by parameters $a_1, a_2, \dots$, and a specialization (α, β, γ). Then, there is a natural Markov process which

8. and conjecturally this family is the largest.

acts on such measures and bring them to similar measure in which the specialization ρ has evolved. The parameter $\gamma \in \mathbb{R}_+$ can be interpreted as the time in the evolution. Likewise, for a single row, there is a similar evolution that acts similarly. Moreover, there exist nice interpretations of the evolution of the array. We will describe only one, in which we are particularly interested.

When the parameters q, t of the Macdonald functions are such that $t = 0$, the functions degenerate to q-Whittaker functions, and we speak about q-Whittaker processes. In that case, when $\alpha = \beta = 0$, we have the equality in law

$$\left(\lambda_n^{(n)}\right)_{n\geqslant 1} = \left(x_n(\gamma) + n\right)_{n\geqslant 1}$$

where

- $\left(\lambda_n^{(n)}\right)_{n\geqslant 1}$ form the left diagonal coordinates of an interlacing triangular array distributed according to the Macdonald process with parameters $a_1, a_2, \dots$ and specialization $(\alpha = 0, \beta = 0, \gamma)$,
- $x_n(\gamma)$ is the location of the nth particle at time $t = \gamma$ in a q-TASEP started from step initial condition, where the nth particle jumps to the right at rate

  $$a_n(1 - q^{x_{n-1}(t) - x_n(t) - 1}).$$

  This is a slight generalization of the q-TASEP presented in Section 1.2.2 since we allow particle-dependent velocities a_n.

This observation explains an algebraic motivation for the introduction of the q-TASEP. As we have already mentioned, when $q = t = 0$, then the Macdonald process degenerates to the Schur process. Actually, the q-TASEP is a marginal of the (time evolution acting on) q-Whittaker process exactly in the same way as the TASEP is a marginal of the Schur process.

There exists a family of operators on functions in n variables which are diagonalized by Macdonald polynomials. By interpreting the eigenvalues as observables and encoding the action of the Macdonald operators as contour integrals, [BC14] provides exact formulas for the moments of $q^{\lambda_n^{(n)}}$. This is a first way of studying the q-TASEP. Proposition 3.1.5 in [BC14] gives the following formula for the observable $q^{k\lambda_n^{(n)}}$, under the Macdonald process with parameters $a_1, a_2, \dots$ and specialization (α, β, γ).

$$\mathbb{E}\big[q^{k\lambda_n^{(n)}}\big] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2i\pi)^k} \int \dots \int \prod_{1\leqslant A<B\leqslant k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{g(q z_j, \rho)}{g(z_j, \rho)} \frac{\mathrm{d}z_j}{z_j}, \tag{1.4}$$

where the contours are *nested*, i.e. the contour for z_j contains all a_i, and all $q z_i$ for $i > j$, but no other singularity, and $g(z, \rho)$ is an (explicit) analytic function of the variable z which depends on the specialization $\rho = (\alpha, \beta, \gamma)$.

For example, if we want to compute $\mathbb{E}\left[q^{k(x_n(t)+n)}\right]$, where the $x_n(t)$ are q-TASEP locations, one can apply Equation (1.4) and the corresponding function g is

$$g(z) = e^{tz} \prod_{i=1}^{n} (z/a_i; q)_\infty.$$

Since the random variables $q^{\lambda_n^{(n)}}$ are in between 0 and 1, the knowledge of their moments identifies the distribution. Moreover, by taking a (q-deformed) generating series of the moments of $q^{\lambda_n^{(n)}}$, one can compute the e_q-Laplace transform (defined in Section 1.2.3), which takes the form of a Fredholm determinant (this is explained in Section 1.4.3).

Theorem 3.2.11 in [BC14] states that under a few technical assumptions that we do not make precise here,

$$\mathbb{E}\left[\frac{1}{\left(\zeta q^{\lambda_n^{(n)}}; q\right)_\infty}\right] = \det\big(I + K_\zeta\big)_{\mathbb{L}^2(C)}, \tag{1.5}$$

where C is a positively oriented contour around 1 and the operator K_ζ is defined in terms of its integral kernel

$$K_\zeta(w, w') = \frac{1}{2i\pi}\int_{-i\infty+\delta}^{i\infty+\delta} \Gamma(-s)\Gamma(1+s)(-\zeta)^s \frac{g(q^s w, \rho)}{g(w, \rho)} \frac{\mathrm{d}s}{q^s w - w'},$$

where the function $g(z, \rho)$ is the same as in (1.4) and δ is a small enough positive real number. The right-hand-side of (1.5) is a Fredholm determinant. We discuss in more detail why Fredholm determinants arise in Section 1.4.3. Moreover, the formula holds for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. In Chapter 2, we perform an asymptotic analysis of this Fredholm determinant, in the case corresponding to the q-TASEP, and we prove a limit theorem for the positions of particles.

1.4.2 The duality method

Let us present another – and more probabilistic – way to recover the nested contour integral moment formulas like (1.4). This approach relies on the combination of two tools: Markov duality and the Bethe ansatz.

Let $\vec{X} = \big(\vec{x}(t)\big)_t$ a continuous-time Markov process on some state-space $\mathbb{X}$, and $\vec{Y} = \big(\vec{y}(t)\big)_t$ another Markov process on a possibly different space $\mathbb{Y}$. We say that $\vec{X}$ and $\vec{Y}$ are dual [EK09, 4.4] with respect to a function

$$H : \mathbb{X} \times \mathbb{Y} \longrightarrow \mathbb{R},$$

if for any initial conditions $\vec{x}(0)$ and $\vec{y}(0)$,

$$\mathbb{E}^{\vec{x}(0)}\big[H(\vec{x}(t), \vec{y}(0)\big] = \mathbb{E}^{\vec{y}(0)}\big[H(\vec{x}(0), \vec{y}(t)\big],$$

where the superscript ${}^{\vec{x}(0)}$ (resp. ${}^{\vec{y}(0)}$) is the initial condition for the process $\vec{x}(t)$ (resp. $\vec{y}(t)$). If we denote by L^X and L^Y the generator of the processes $\vec{X}$ and $\vec{Y}$, then the duality is equivalent to

$$L^X H = L^Y H,$$

where L^X acts on the $\vec{x}$ variable and L^Y acts on the $\vec{y}$ variable.

In the discrete-time setting, it is more common to speak about intertwining. A matrix $H : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ intertwines two Markov transition kernels $P_{\vec{X}}$ and $P_{\vec{Y}}$ if

$$P_{\vec{X}} H = H\big(P_{\vec{Y}}\big)^T,$$

where ^{T} denotes the transpose. In words, it means in both cases that the action of the generator (resp. transition kernel) of the process $\vec{X}$ on H is the same as that of the process $\vec{Y}$. There are two situations where a duality is clearly useful [JK14]:

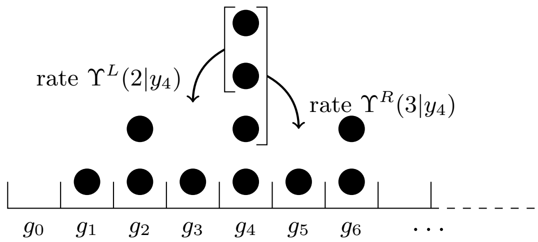

Figure 1.10: A general zero-range process.

- When one of the processes is very difficult to analyse, whereas the other one is simple. In such cases, one cannot expect that the duality functional characterizes much the complicated process, but it can produce useful results.
- When the functional H characterizes the system, for instance when the knowledge of $\mathbb{E}^{\vec{x}(0)}\big[H(\vec{x}(t), \vec{y}(0)\big]$ for any $\vec{y}(0)$ determines uniquely the distribution of $\vec{x}(t)$.

Exclusion processes vs Zero-Range

One can generally build a coupling between exclusion processes on $\mathbb{Z}$ and zero-range processes. We consider here zero-range processes on N sites, say $\{0, 1, \dots, N-1\}$, and N might be infinite. Above each site, there is a certain number of particles that may move to one of the neighbouring sites.

Let us describe informally the particle dynamics in the continuous time setting. The family of processes that we describe is slightly more general that what one usually considers as zero-range processes. Assume that $g_i(t)$ particles are above site i at time t. Then,

- $j \leqslant g_i(t)$ particles move together to the right to site $i+1$ with exponential rate $\Upsilon^R\big(j|g_i(t)\big)$,
- and $j' \leqslant g_i(t)$ particles move to the left to site $i-1$ with rate $\Upsilon^L\big(j|g_i(t)\big)$.

All sites are updated independently.

It happens that if $\big(x_n(t)\big)$ is a continuous-time exclusion process described as in Section 1.2.1 by rates $\phi^R(j|m)$ and $\phi^L(j|m)$, then the dynamics of the gaps between particles, i.e.

$$g_i(t) := x_{i-1}(t) - x_i(t) - 1,$$

have zero range dynamics, and the rates are the same as those of the exclusion process:

$$\forall j \leqslant m, \begin{cases} \Upsilon^R(j|m) & = \phi^R(j|m), \\ \Upsilon^L(j|m) & = \phi^L(j|m). \end{cases}$$

This coupling is exact for processes on the ring $\mathbb{Z}/N\mathbb{Z}$. In infinite volume with an infinite number of particles one has to be careful with what happens on the boundaries, but couplings can certainly be written down precisely.

Example: duality method for q-TASEP

We describe now a method to recover Macdonald moment formulas for q-TASEP by exploiting a duality between the q-TASEP and its associated zero-range process. This method was developed for the q-TASEP in [BCS14], and uses some ideas that were already present in [IS11]. The method works with a few steps. It is important to notice that the same steps lead to the exact solvability of discrete time q-TASEP [BC13], the q-Hahn TASEP [Cor14] and the asymmetric q-Hahn TASEP [BC15a] (see also Chapter 3).

1. **Duality:** Let us call q-totally asymmetric zero-range process (q-TAZRP) the zero range process which is naturally associated to the q-TASEP. Define

$$H(\vec{x},\vec{g}) = \prod_{i=0}^{N} q^{g_i(x_i+i)},$$

with the convention [9] that $H = 0$ if $g_0 > 0$. By applying the generators of the q-TASEP and the q-TAZRP to the function H, one readily sees that the q-TASEP with N particles and the q-TAZRP on $\{0,1,\dots N\}$ are dual with respect to H.

2. **True evolution equation:** For a given configuration of N particles $\vec{x}$, we consider the function

$$u(\vec{g},t) = \mathbb{E}^{\vec{x}}\left[H(\vec{x}(t),\vec{g})\right],$$

where $\vec{x}(t)$ is the vector of the first N particles of a q-TASEP at time t started from initial condition $\vec{x}$. We denote L^{qTASEP} and L^{qTAZRP} the generators of the q-TASEP and the q-TAZRP. By Kolmogorov backward equation,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L^{qTASEP}\mathbb{E}^{\vec{x}}\left[H(\vec{x}(t),\vec{g})\right].$$

Using the commutation between the generator and the semi-group of the q-TASEP,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbb{E}^{\vec{x}}\left[L^{qTASEP}H(\vec{x}(t),\vec{g})\right].$$

Using the duality, and then the commutation between the generator and the semi-group of the q-TAZRP, we arrive at

$$\forall\vec{g},\ \frac{\mathrm{d}u}{\mathrm{d}t} = L^{qTAZRP}\mathbb{E}^{\vec{x}}\left[H(\vec{x}(t),\vec{g})\right]. \tag{1.6}$$

Equation (1.6) is called the *true evolution equation*, it is a system of infinitely many ODEs. In the case of q-TASEP, the solution is unique because the system of equations is triangular. More generally, one has to restrict the class of functions u considered in order to prove the uniqueness of the solution.

3. **Bethe ansatz:** First, one makes a change of variables and describe the q-TAZRP configurations using ordered particles locations instead of the coordinates of $\vec{g}$. More precisely, instead of writing $u(\vec{g},t)$, we write $u(\vec{n},t)$ where

$$\vec{n}\in\mathbb{W}^k := \{n_1 \geqslant n_2 \geqslant \dots \geqslant n_k : n_i \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant i \leqslant k\},$$

and the n_is are particle locations. The integer k is the number of particles, which is conserved by q-TAZRP dynamics. Hence, the true evolution equation is an equation for functions

$$f : \mathbb{W}^k \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}.$$

9. We recall that we still use the convention that there is a virtual particle at $x_0 = +\infty$ in our description of exclusion system.

Using a variant of the *Bethe ansatz*, one is sometimes able to show that the solution to the true evolution equation solves a simpler system of equation on a larger functional space, provided some *boundary condition* is satisfied on the boundary of the physically meaningful space. In the example of the q-TASEP, the larger space is the space of functions

$$f : \mathbb{Z}^k \times \mathbb{R}_{\geqslant 0} \to \mathbb{R},$$

and the boundary condition must hold for $\vec{n}$ on the boundary of $\mathbb{W}^k$ (i.e. when $\vec{n}$ have at least two coordinates equal). The simpler system of equations corresponds to the dynamics of non-interacting particles, and hence is called *free evolution equation.* The solution to the free evolution equation + boundary conditions coincide with the solution of the true evolution equation on the small functional space (here functions $\mathbb{W}^k \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$). This step is called an "ansatz" since it does not work in general. It works only for the particular exactly solvable models that we consider. In the examples of Chapters 3 and 4, the equivalence of true and free evolution equations is proved using non-commutative binomial formulas, which are generalizations of Lemma 1.2.1. In this algebraic setting, one interprets the boundary condition as a commutation relation.

4. **Moment formulas:** One can search solutions to the free evolution equation with boundary conditions in the form of nested contour integral formulas as (1.4). It turns out that the boundary conditions can be checked easily on such formulas. Once we have solved the sytem, we are able to write all the q-moments $\mathbb{E}[q^{kx_n(t)}]$, which determine the distribution of $x_n(t)$.

5. **Fredholm determinant:** Using the tools developed for Macdonald processes, one is able to form the q-moment generating series and one gets a Fredholm determinant formula for the e_q-Laplace transform of the variable $q^{x_n(t)}$.

6. **Conclusion:** The e_q-Laplace transform can be inverted to recover the distribution of $x_n(t)$ [BC14, Proposition 3.1.1]. In practice, this step is often unnecessary.

1.4.3 Fredholm determinant formulas

Introduction to Fredholm determinants

All the models that we have defined previously admit closed formulas for the Laplace transform – or its q-deformed counterpart – of meaningful observables, in terms of Fredholm determinants. As we explain in Section 1.5, these formulas are convenient to prove limit theorems towards the Tracy-Widom distribution. However, we do not need to use any of the properties of the theory of Fredholm determinants. The only thing that we need is an explicit expression, in the form of a Fredholm determinant. We provide here a superficial introduction to Fredholm determinants, following [BC14, Definition 3.2.6], and we only explain how to compute them numerically. We refer to the chapter 3 of [Sim79] for a more complete introduction of Fredholm determinants.

Definition 1.4.1. Fix a Hilbert space $\mathbb{L}^2(X, \mu)$, where X is a measure space and μ a measure on X. In this manuscript, X is often a contour in the complex plane. When $X = \Gamma$ is a simple positively oriented smooth contour in $\mathbb{C}$, we write only $\mathbb{L}^2(\Gamma)$ and μ is implicitly understood to be the path measure along Γ divided by $2i\pi$.

Let K be an integral operator, i.e. an operator acting on functions $f \in \mathbb{L}^2(X, \mu)$ by

$$Kf(x) = \int_X K(x, x')f(x')\mathrm{d}\mu(x').$$

The function $K : X^2 \to \mathbb{C}$ is called the kernel of K. The Fredholm determinant of $I + K$ is defined by

$$\det(I + K)_{\mathbb{L}^2(X)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \dots \int_X \det\left[K(x_i, x_j)\right]_{i,j=1}^n \prod_{i=1}^n \mathrm{d}\mu(x_i).$$

A sufficient condition for the above series to converge is that the operator K is trace-class. In this manuscript, we manipulate kernels with explicit expressions, and we can always check directly that the Fredholm determinant expansions are absolutely convergent.

From moment formulas to Fredholm determinants

A general scheme for going from nested contour integral moments formulas to Fredholm determinant has been developed to study the q-Whittaker process in [BC14, Section 3.2.2]. It turns out that this method is fruitful beyond Maconald processes and can be applied for other systems such as the q-Hahn TASEP, the q-Hahn asymmetric exclusion process, and their scaling limits. It is natural to expect that Fredholm determinants arise in the study of TASEP or directed last passage percolation with geometric weights, because these processes can be seen as determinantal point processes. However, Fredholm determinant can arise far beyond determinantal point processes. We describe here the case of exclusion processes for which one can apply the duality method.

Assume that we have a moment formula for observables $q^{x_n(t)}$, where $x_n(t)$ are particles position of a system that we do not specify since the methods are general. If we get this formula from the duality method explained in Section 1.4.2, it will have the same form as for the observables of the q-Whittaker process in Equation (1.4), i.e.

$$\mathbb{E}[q^{x_n(t)}] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2i\pi)^k} \int \dots \int \prod_{1 \leqslant A < B \leqslant k} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^k \frac{g(qz_j)}{g(z_j)} \frac{\mathrm{d}z_j}{z_j},$$

for some analytic function g (analytic outside finitely many singular points), where the contour for z_j contains a fixed set of singularities, and all qz_i for $i > j$. In most cases we deal with in this thesis, the fixed set of singularities in simply $\{1\}$.

The first step is to write this formula using the same contour for each variable. We have

$$\mathbb{E}\left[q^{kx_n(t)}\right] = [k]_q! \sum_{\lambda \vdash k} C(\lambda) \int \dots \int \det\left(\frac{1}{w_i q^{\lambda_i} - w_j}\right)_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_j} \frac{g(w_j)}{g(q^{\lambda_j} w_j)} \mathrm{d}w_1 \dots \mathrm{d}w_{\ell(\lambda)}, \tag{1.7}$$

where the integration contour is a small circle around 1 (more generally a fixed set of singularities) excluding all other singularities. The sum runs over integer partitions $\lambda \vdash k$ where $\ell(\lambda)$ is the number of non-zero components, and $C(\lambda)$ is an explicit constant depending on λ that we do not specify here. This type of transformation is called the *contour shift argument* (see [BCPS15, Proposition 7.4]) In order to prove this, one needs to shrink all contours to a small circle around 1. During the deformation of contours, one encounters all poles of the product

$$\prod_{A<B} \frac{z_A - z_B}{z_A - q z_B}.$$

One can naturally associate an integer partition to each residue. Then, the proof amounts to group the residues corresponding to the same partition, and factor their contribution using the Cauchy determinant formula and symmetrization identities. The form (1.7) is very useful for computing the e_q-Laplace transform of $q^{x_n(t)}$. Indeed,

$$\mathbb{E}\big[e_q(\zeta q^{x_n(t)})\big] = \sum_{k=0}^{\infty} \frac{\zeta^k \mathbb{E}\left[q^{k x_n(t)}\right]}{[k]_q!}.$$

When $x_n(t)$ corresponds to an interacting particle system started from step initial condition, $q^{x_n(t)}$ is bounded, and hence the exchange between summation and expectation is valid for $|\zeta|$ small enough. Under some assumptions on the function g, this allows one to write a Fredholm determinantal fromula for $\mathbb{E}\big[e_q(\zeta q^{x_n(t)})\big]$. We generally find that

$$\mathbb{E}\big[e_q(\zeta q^{x_n(t)})\big] = \det(I + K_\zeta)_{\mathbb{L}^2(C_1)},$$

where C_1 is a positively oriented small contour around 1, and K_ζ is defined by its integral kernel

$$K_\zeta(w, w') = \frac{1}{2i\pi} \int_{-i\infty+\delta}^{i\infty+\delta} \Gamma(-s)\Gamma(1+s)(-\zeta)^s \frac{g(q^s w)}{g(w)} \frac{\mathrm{d}s}{q^s w - w'}, \tag{1.8}$$

where δ is a small enough positive real number. As in Section 1.4.1, this holds for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. We have already explained that formula (1.8) holds for the q-TASEP. It also holds for the q-Hahn TASEP [Cor14] and for the q-Hahn asymmetric q-TASEP [BC15a] with slightly more complicated expresssions for g (see Theorem 3.3.13).

Moreover, the Fredholm determinantal formula of the q-Hahn TASEP has a limit when q goes to 1, which corresponds to the Laplace transform of the random variable

$$\mathsf{P}\big(X_t \geqslant x\big),$$

where X_t is a Beta-RWRE. By taking a further limit corresponding to the zero temperature limit, one also finds a Fredholm determinantal formula for the probability distribution function of the passage times in the Bernoulli-Exponential FPP model.

1.5 Limit Theorems

In this section we describe the main results of this thesis, which are limit theorems. We first explain the limit theorems predicted for interacting particle systems in the KPZ universality class, and provide some background on the Tracy-Widom distribution and BBP phase transition from random matrix theory. Then, we state our main results.

1.5.1 KPZ scaling theory

The KPZ scaling theory constitutes an educated guess to compute exactly the expressions of all constants arising in limit theorems for exclusion processes in the KPZ universality class. The range of applicability of this theory is actually larger than exclusion processes. However, it comes from the physics literature, in particular the works of Krug, Meakin and Halpin-Healy [KMHH92], and the results are so far highly conjectural. In this section, we present these heuristic claims in the case of exclusion processes, following the approach of Spohn [Spo12]. We try to focus on intuitions, and warn the reader that some statements would need more precise definitions to be rigorous.

Consider an exclusion process as presented in Section 1.2. Assume that the translation invariant stationary measures are precisely labelled by the density of particles ρ, where

$$\rho = \lim_{n\to\infty} \frac{1}{2n+1} \#\{\text{particles between } -n \text{ and } n\}.$$

The large-time behaviour of the process is determined by the type of initial condition and two quantities that summarize the dynamics:

- **The average steady-state current.** Denoted $j(\rho)$, it is the expected number of particles going from site 0 to 1 per unit time, for a system distributed according to the stationary measure indexed by ρ.
- **The integrated covariance.** Denoted $A(\rho)$, it corresponds to the size of fluctuations of the macroscopic density. It is defined by

  $$A(\rho) = \sum_{j\in\mathbb{Z}} \mathrm{Cov}(\eta_0, \eta_j),$$

  where $\eta_0, \eta_j \in \{0,1\}$ are the occupation variables of the exclusion system at sites 0 and j, and the covariance is taken under the ρ-indexed stationary measure.

Let us precise first the hydrodynamic limit of the system. One expects that the rescaled particle density $\varrho(x,\tau)$, given heuristically by

$$\varrho(x,\tau) = \lim_{\tau\to\infty} \mathbb{P}(\text{There is a particle at } \lfloor xt \rfloor \text{ at time } t\tau)$$

satisfies the conservation equation

$$\frac{\partial}{\partial\tau}\varrho(x,\tau) + \frac{\partial}{\partial x} j(\varrho(x,\tau)) = 0, \tag{1.9}$$

with initial condition which is $\varrho(x,0) = \mathbb{1}_{x<0}$ for the step initial condition. This type of PDE (conservation equation) can be solved using the characteristics method. The so-called entropy solution, which corresponds to the physically meaningful solution, is given by a variational principle. The solution of this PDE yields a law of large numbers for the position of particles. For $\kappa \geqslant 0$, if n and t go to infinity with $n = \lfloor \kappa t \rfloor$, then one has

$$\frac{x_n(t)}{t} \xrightarrow[t\to\infty]{a.s.} \pi(\kappa).$$

It turns out that instead of expressing π as a function of κ, it is more convenient to parametrize π and κ by the local density ρ. The existence of such a parametrization is given by the solution of the PDE (1.9): $\pi(\rho)$ is implicitly determined by $\rho = \varrho(\pi(\rho), 1)$ and $\kappa(\rho)$ is determined by $\pi(\kappa(\rho)) = \pi(\rho)$.

Let $\lambda(\rho) = -j''(\rho)$ and assume that $\lambda(\rho) \neq 0$. This hypothesis is very important, it corresponds to the fact that the non-linearity does not vanish in the limit to the KPZ equation. Then, for such a ρ, the KPZ class conjecture states that (starting from step initial condition) we expect the convergence in distribution

$$\frac{x_n(t) - t\pi(\rho)}{\sigma(\rho)t^{1/3}} \Longrightarrow \mathcal{L}_{GUE}, \tag{1.10}$$

where the constants π, κ and σ are given by

$$\pi(\rho) = \frac{\partial j(\rho)}{\partial \rho}, \quad \kappa(\rho) = -\rho \frac{\partial j(\rho)}{\partial \rho} + j(\rho), \quad \sigma(\rho) = \left(\frac{\lambda(\rho)\big(A(\rho)\big)^2}{2\rho^3} \right)^{1/3};$$

$\mathcal{L}_{GUE}$ is the Tracy-Widom GUE distribution introduced in Section 1.5.2, and n goes to infinity with $n = \lfloor \kappa(\rho)t \rfloor$. The expression for $\pi(\rho)$ is a consequence of the conservation equation. Then, the expression for $\kappa(\rho)$ can be inferred by simple arguments, and it depends on the initial condition (here we consider step initial condition).

Digression 1.5.1. *Why this expression for $\sigma(\theta)$? The role played by the integrated covariance can be simply explained. This is just the analogue of the variance in limit theorems for sums of correlated random variables. Indeed, assume that $(g_i)_{i\in\mathbb{Z}}$ is a stationary sequence of random variables of mean g. Let $x_n = \sum_{i=1}^n g_i$. Then, if the sequence g_i is not too correlated, we expect a central limit theorem, i.e.*

$$\frac{x_n - gn}{\sigma\sqrt{n}} \Longrightarrow \mathcal{N}(0,1).$$

The coefficient σ^2 must be the limit of the variance of x_n divided by n, and using stationarity, it is readily calculated

$$\sigma^2 = \sum_i \mathrm{Cov}(g_0, g_i). \tag{1.11}$$

One should think that the letter g here stands for "gap", and the position of a particle can be realized as a sum of gaps. This explains why the fluctuations of particle locations should rescale by the integrated covariance. However, there is not much intuition behind the factor λ. We shall only mention that in terms of height function, the limit shape of the process is the convex conjugate of the function j, so that λ can be interpreted as the radius of curvature of the limit shape.

The first paper [KPZ86] about KPZ universality class introduced the KPZ equation and predicted only fluctuation exponents. Krug, Meakin and Halpin Healy [KMHH92], by analyzing a few systems in the KPZ universality class both numerically and with respect to their limit to the KPZ equation, where able to predict the expression for $\sigma(\rho)$. The law of large numbers have been known for much longer time. However, it is only from the works of Johansson about the fluctuations of passage times in directed last passage percolation[10] [Joh00], that the community has expected the $\mathcal{L}_{GUE}$ to arise in general. The KPZ scaling theory is actually more complete than what we presented. [KMHH92] also predicts how to renormalise the space coordinate in order to see a non trivial limit at the level of processes. Starting from step initial condition, the Airy process is expected to arise in general [PS02].

So far, the limit theorem (1.10) can be proved only for exactly solvable models.

10. This model can be coupled with a discrete-time TASEP.

1.5.2 Tracy-Widom distribution and the BBP phase transition

Almost all the results that we state in this section are limit theorems towards the Tracy-Widom distribution. Two different approaches [TW94] can be used in order to define this distribution and both can be used for proving limit theorems [11]. On one hand, one can give an exact formula for the probability distribution function, in terms of the Painlevé II nonlinear ODE. On the other hand, one can define the Tracy-Widom distribution as the scaling limit of extreme points of determinantal point processes.

We follow the second approach. The Tracy-Widom distribution is defined as the distribution of the right-most point of a point process in $\mathbb{R}$ defined by its n point correlation functions

$$\rho_n(x_1, \dots, x_n) = \det\Big(K(x_i, x_j)\Big)_{i,j=1}^{n},$$

where the determinantal kernel K is defined by

$$K(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y},$$

and Ai is the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2i\pi}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} e^{z^3/3 - zx}\mathrm{d}z.$$

By the inclusion-exclusion principle, the distribution function of the right-most point of such a determinantal point process can be written in terms of a Fredholm determinant. Hence, we have

$$F_{\mathrm{GUE}}(x) = \det(I-K)_{\mathbb{L}^2(x,+\infty)}.$$

We use the subscript GUE because this distribution function has been discovered by Tracy and Widom as the limit distribution of the rescaled largest eigenvalue of a Gaussian Hermitian random matrix (such matrices are said to belong to the *Gaussian Unitary Ensemble* abbreviated GUE), when the size of the matrix goes to infinity. The top eigenvalue fluctuates on the $n^{1/3}$ scale, where n is the size of the matrix.

If one adds a perturbation X to a matrix M of the GUE, then the top eigenvalue of $M + X$ does not necessarily fluctuate on the $n^{1/3}$ scale with Tracy-Widom fluctuations. Actually, there is a phase transition, depending on the perturbation eigenvalues and the rank of the perturbation X [Péc06], between a Gaussian behaviour and a "Tracy-Widom" behaviour. At the critical point of this phase transition, the top eigenvalue fluctuates on the $n^{1/3}$ scale. However, the limiting law is not the Tracy-Widom distribution but a perturbation of it that we call the BBP distribution. This distribution have been introduced in [BBAP05] in a slightly different context, namely perturbation of covariance matrices of Gaussian samples. It can be defined like the Tracy-Widom distribution, using a determinantal kernel which is a perturbation of the Airy kernel. The probability distribution function can also be written in terms of Painlevé equations [Bai06].

The interesting feature of the Tracy-Widom distribution is its universality. There does not exist any convenient theorem that delineates the cases where the Tracy-Widom distribution should appear, but the study of quite diverse probabilistic models advocates for an

11. There exists other characterizations of the Tracy-Widom distribution, for instance in [RRV11], but it is not clear if these are useful in the context of KPZ universality.

extraordinary ubiquity. The BBP distribution also shows up in seemingly unrelated models featuring a phase transition between Gaussian and Tracy-Widom (GUE) behaviour. It is universal to some extent, in the sense that it does not depend on the details of the model, and appear in the asymptotic analysis in the following rather general case. Assume that the fluctuations are characterized by the Fredholm determinant of an operator with a kernel which rescales to the Airy kernel. Imagine that the kernel has an integral representation, like the Airy kernel. The presence of a pole in the integrand located away from the critical point (through which the integration contour will have to go, during the steepest-descent analysis) will not change much the asymptotic behaviour. However, if one can play with the position of the pole as a parameter, then one observes the BBP transition when the pole goes through the critical point. The transition of order k is obtained with a pole of order k. To the best of the author's knowledge, this phenomenon is so far understood only at the heuristic level.

We have already mentioned in Section 3.4 about KPZ scaling theory that we expect that the rescaled positions of particles for exclusion processes in the KPZ universality class are Tracy-Widom distributed. For the TASEP – and as we see very soon, also for the q-TASEP – the BBP phase transition occurs when one varies the density of particles in the initial condition, or when one adds slower particles in the system (see [Bai06] and references therein).

1.5.3 q-TASEP with slower particles

For the q-TASEP, the KPZ scaling conjecture has been shown by Ferrari and Vető in [FV13] for one-point fluctuations, under a restriction on the parameters. Actually, the result is proved only in the part of the rarefaction fan where the speed of particles is below 1/2 (it means that a macroscopic part of the rarefaction fan was missing). In [Bar15], we extend [12] Ferrari-Vető's proof to remove this restriction, and also study the case when all but finitely many particles have speed 1 (i.e. they jump at rate $1 - q^{gap}$) and a finite number of particles have a slower speed (they jump at rate $\alpha(1 - q^{gap})$ for some $\alpha < 1$). The case of a faster particle is not interesting, since it has no influence on the macroscopic scale. According to the parallel with the TASEP, it is reasonable to expect to see the BBP transition, and this is proved in [Bar15]. More precisely, the largest eigenvalue of the deformed GUE corresponds to the inverse of the speed α, whereas the multiplicity of this eigenvalue corresponds to the number of particles having the minimal speed α.

To understand this intuitively, let us examine the case where only the first particle is slower and has rate α. It is quite evident that the first particle will create a traffic jam: by coupling the slowed-down process with a usual qTASEP, one can see that the second particle (and more generally the nth particle) would like to move faster than α in average, but it cannot since the first particle blocks. Hence, in the presence of a slower particle, the next particles are slowed down, as in a traffic jam. How long is the traffic jam ? Is it macroscopic ? One can provide heuristic answers to these questions. The particles concerned by the traffic jam are those which, in absence of a slower particle, would have an average speed greater than α. This concerns a macroscopic quantity of particles. What the BBP transition says then – and this cannot be inferred from simple probabilistic

12. This extension is mostly technical: The difficulty is located in the study of a particular analytic function $\mathbb{C} \to \mathbb{C}$ which turns out to determine the asymptotic behaviour. This function involves q-deformed digamma functions, and hence is not easy to study precisely.

arguments [13] – is that inside the traffic jam, particles have a Gaussian behaviour [14], outside the traffic jam, they behave exactly as in absence of a slower particle, and at the border of the traffic jam, the positions of particles fluctuate according to the BBP distribution of rank 1 (because there is only one slow particle in this example). The Figure 1.11 summarizes these explanations.

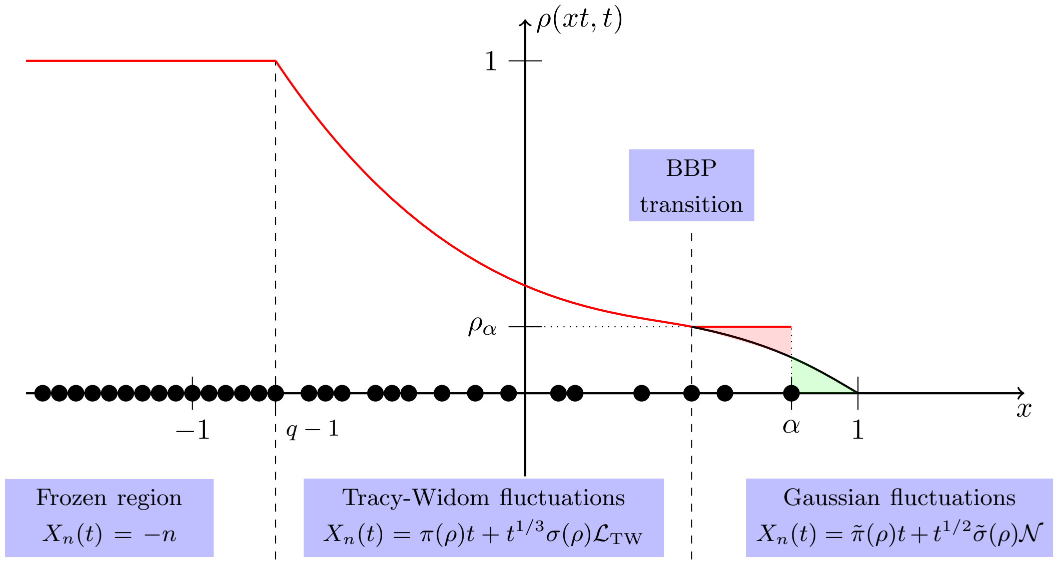

Figure 1.11: Limiting density profile for the q TASEP where the first particle has a slower rate $\alpha < 1$. Note that the area of the gray and the red regions are necessarily equal, if we believe that particles outside the traffic jam does not feel the slower particle (This is proved in [Bar15] and Chapter 2).

1.5.4 Beyond KPZ scaling theory ?

In the context of exclusion processes, the KPZ scaling theory was first verified for the TASEP and Johansson [Joh00] discovered that the fluctuations were Tracy-Widom distributed. Then, it took approximately eight years to extend the result to the case where particles can move to the left and to the right, i.e. for ASEP. It has been done by Tracy and Widom in a series of papers [TW08b, TW08a, TW09], using very clever new arguments at several stages of the proof. With the development of both the theory of Macdonald processes [BC14] and the duality method [BCS14], the introduction of new exaclty solvable exclusion process has accelerated, and all these processes can be solved in a similar way. Thus the KPZ scaling theory was proved [15] for the q-Hahn TASEP in [Vet14]. It was proved for the MADM exclusion process in [BC15a], and a proof was sketched for any asymmetric q-Hahn exclusion process in [BC15a].

13. The position of the critical point can be inferred from simple arguments, for instance the equality of areas in Figure 1.11 but the precise nature of fluctuations cannot.

14. Actually, it is a bit more complicated, since when several particles, say k have the slowest speed α, then the fluctuations are not Gaussian but that of the top eigenvalue of a $k \times k$ Gaussian unitary matrix.

15. with restrictions on the parameter space, and only for one-point fluctuations.

Now that we have enough models at our disposal to be persuaded of the validity of the KPZ scaling conjecture, we can use these exactly solvable models to understand the questions that are not covered by KPZ universality predictions. One question we have been interested in is the statistics of the first particle position. When the process is totally asymmetric (i.e. particles jump only to the right), the central limit theorem closes the question, and there's not much to say. The real question is what happens for partially asymmetric systems. For ASEP, Tracy and Widom showed that the positions of the first particles in ASEP fluctuate on a $\sqrt{time}$ scale. However, the limit is not Gaussian, and the precise statistics are described by a seemingly new probability distribution, defined in terms of the Mehler kernel [TW09].

Apart from ASEP, the only partially asymmetric exclusion system for which one has formulas amenable for precise asymptotic analysis is the asymmetric q-Hahn exclusion process. The formulas are quite complicated in the general parameter case, but it happens that when $\nu = q$, that is the case corresponding to the MADM exclusion process, the formulas simplify. In that case, we know explicit constants σ and π such that as t goes to infinity,

$$\frac{x_1(t) - \pi t}{\sigma t^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

where $x_1(t)$ is the position of the first particle at time t in the MADM exclusion system, and $\mathcal{L}_{GUE}$ is the GUE Tracy-Widom distribution.

To the author's knowledge, there does not exist any general criteria predicting the order of magnitude of the fluctuations of the first particle, for exclusion processes in the KPZ universality class. However, one can reasonably conjecture that the $t^{1/3}$ or $t^{1/2}$ behaviour depends on whether the density profile is continuous at the right-most end of the density profile. A continuous density profile means that around the first particles, the average density is zero, and the distance between particles goes to infinity; whereas a discontinuous density profile means that around the first particles, the density is positive, and equivalently the distance between the first and second particles is bounded by a constant with very high probability. It turns out that the density profile is continuous for ASEP and discontinuous for the MADM, and this can be understood by elementary probabilistic reasoning about the transition rates. However, the precise nature of the fluctuations cannot be understood by elementary arguments, although one can argue that the scale of fluctuations should be related to the strength of interactions, which depends on the typical distance between consecutive particles.

1.5.5 Second order corrections to the large deviation principle for the Beta-RWRE

Now, we explain our results about directed lattice paths models. The form of the next limit theorems, and the techniques used to prove them, are very similar with what we have explained in the case of exclusion processes. However, since we are dealing with completely different models, the probabilistic meaning of the results is quite different.

Consider X_t a space-time random walk in Beta environment (defined in Section 1.3.1). As we have already observed, the model can be seen as a directed polymer with non i.i.d. weights on edges, and the large deviation rate function plays a role analogous to the free energy. More precisely, we know from [RASY13] that the limit

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log\left(\mathsf{E}\left[e^{zX_t}\right]\right)$$

exists $\mathbb{P}$-almost surely. Moreover, the random walk obeys a large deviation principle

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\mathsf{P}\big(X_t > xt\big)\right) = -I(x),$$

where the rate function $I(x)$ is the Legendre transform of λ. This result is quite general for random walks in random environment, provided the environment is mixing enough. In the case of the Beta-RWRE, using a Fredholm determinant representation for the Laplace transform of $\mathsf{P}\big(X_t > xt\big)$, we are able to compute explicitly the rate function $I(x)$, and also prove second order corrections. Under some hypotheses stated in Theorem 4.1.15 in Chapter 4 – although we expect the result to hold quite generally – we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\log\left(\mathsf{P}\big(X_t > xt\big)\right) + I(x)t}{t^{1/3}\sigma(x)} \leqslant y\right) = F_{\mathrm{GUE}}(y), \tag{1.12}$$

where $\sigma(x)$ and $I(x)$ are explicit functions.

A large deviation principle may seem precise enough to characterize the tail behaviour of X_t. This is a reasonable objection if one looks only at X_t. Then, what is the purpose of proving the limit (1.12) ? In general, when one looks at the maximum of a sequence of N independent random variables, for N going to infinity, one knows that the maximum must correspond to a rare event, i.e. an event that would occur with probability of order $1/N$ for a single random variable. This general idea is still true for correlated sequences, with the caveat that positive correlations make the maximum decrease [16], as if we were considering less random variables. Let us consider N random walks

$$\big(X_t^{(1)}\big)_t, \dots, \big(X_t^{(N)}\big)_t,$$

drawn independently in the same environment. These random walks are of course strongly correlated by the fact that they share the same environment. According to the heuristics that we have just mentioned, the maximum of the sequence $X_t^{(1)}, \dots, X_t^{(N)}$ should be "an event of probability $\mathcal{O}(1/N)$". Thus, if we set $N = e^{ct}$ for some constant c, then we know that the maximum is in the large deviation region. This is why large deviations are related to statistics of extrema. Moreover, the large deviation principle would give only the first order of the maximum. The second order of the maximum is accessible only via second order corrections to the large deviation principle. This explain an application of the limit (1.12): We show in Corollary 4.5.8 that as t goes to infinity, the maximum of the sequence $X_t^{(1)}, \dots, X_t^{(N)}$, properly rescaled, converge to the Tracy-Widom distribution.

What is maybe even more interesting is that we are able to describe very precisely the (asymptotic) covariance structure of that sequence (see Proposition 4.5.12). We also know the asymptotic law: X_t weakly converges to a Gaussian random variable when rescaled by $\sqrt{t}$. Hence, the model of N independent Beta-RWRE in the same environment can be seen as a toy model for studying extrema of strongly correlated sequences. However, we know from old results from the asymptotic theory of extreme value statistics (see Section 4.5.3, and more generally the book of Galambos [Gal87]) that the maximum of a Gaussian vector with the same covariance structure satisfies a quite different limit theorem.

16. Slepian's lemma constitutes a rigorous formulation of this claim in the particular Gaussian case.

1.5.6 Bernoulli-Exponential FPP

It is remarkable that the limit theorem towards the Tracy-Widom distribution proved for the Beta-RWRE propagates at zero temperature. Let us explain the correspondence between transition probabilities of the Beta-RWRE and passage times of the Bernoulli exponential first passage percolation model. Define the map

$$\mathbf{T} : p \mapsto -\epsilon \log(p),$$

where ϵ plays the role of the temperature which goes to zero, and p is a probability. This map transforms products of probabilities into a sum of passage times, modulo a rescaling by ϵ. Hence, the probability of a random walk path π can be written in terms of passage times as

$$\exp(\epsilon^{-1} \sum_{e \in \pi} t_e),$$

where the t_e correspond to the passage times of edges along the path π. In the limit $\epsilon \to 0$, if P is the probability of an ensemble of paths, then $\mathbf{T}(P)$ will converge to the minimum passage time of these paths. This is why the limit theorem for

$$\frac{1}{t} \log\Big(\mathsf{P}\big(X_t > xt\big)\Big)$$

as t goes to infinity in the Beta-RWRE model should correspond to a limit theorem for

$$\frac{T(n, \kappa n)}{n}$$

as n goes to infinity, where $T(n, m)$ is a first-passage time (to the half line $D_{n,m}$, see Section 1.3.3) in the FPP model. This limit theorem is stated and proved in Section 4.6.

Open questions

As a conclusion, we would like to discuss a few perspectives of future research raised by the results from this thesis.

- In the study of the q-Hahn asymmetric exclusion process, we have seen that the fluctuations of the first particle were Tracy-Widom distributed on a cube-root scale, contrasting with the situation for ASEP. One can imagine that this is due to the fact that the density around the first particle is positive, resulting in stronger attraction/repulsion phenomena. One can find in the physics literature other processes which exhibit such discontinuous density profiles, for instance facilitated exclusion processes [GKR10], and a sketchy analysis suggests that the fluctuations are on the cube-root scale but not GUE-distributed. However, this could be an artefact due to the very particular dynamics of facilitated exclusion processes. In order to make the situation more clear, one would need to study other exactly solvable examples, and see if universal rules seem to emerge.
- Another question related to the q-Hahn asymmetric exclusion processes concerns its scaling limits. We know that after an appropriate rescaling, some observables of the q-TASEP converge to the partition function of the O'Connell-Yor semi-discrete polymer. What is the corresponding limit of the two-sided q-TASEP ?

- The eigenfunctions of the generator of the q-Hahn asymmetric exclusion process are actually the same as for the q-Hahn TASEP. Hence, the underlying algebraic structure is essentially the same. These eigenfunctions are rational deformations of Hall-Littlewood symmetric functions. However, a generalization of these eigenfunctions have been recently studied in [Bor14] and the corresponding interacting particle systems have been introduced in [CP15]. It would be interesting to see if two-sided generalizations of these processes can be defined and studied as well.
- The Beta random walk in random environment is the first example of a random walk in random environment exhibiting KPZ behaviour. This opens many questions. Should we expect similar results for a wide class of random walks in random environment? How one should extend the framework of KPZ universality in order to encompass RWRE? Moreover, the logarithm of the quenched probability distribution of the Beta RWRE should be understood as a free energy, and the results we have proved are Tracy-Widom fluctuations for the free energy. In light of universality predictions for directed polymers and in particular the KPZ relation between fluctuation exponents, one could expect that conditionally on a large deviation event, the endpoint of the Beta-RWRE fluctuates on a $n^{2/3}$ scale.
- A much deeper question concerns the Tracy-Widom distribution. In many cases, limit theorems towards the Tracy-Widom distribution deal with the maximum – or minimum– of many correlated random variables. For example, the eigenvalues of a GUE matrix, the passage times associated to directed paths in last passage percolation, the endpoints of many Beta-RWRE paths in the same environment, are all sequences of correlated random variables. What should be the hypotheses on a sequence of correlated random variables, in order to reasonably expect Tracy-Widom fluctuations for the maximum?

Chapter 2

Asymptotic analysis of the q-TASEP

This chapter corresponds to the paper:

[Bar15] G. Barraquand, *A phase transition for q-TASEP with a few slower particles*, Stochastic Process. Appl. **125** (2015), no. 7, 2674 – 2699.

Contents

2.1 Introduction and main result

The totally asymmetric simple exclusion process is a stochastic model of particles on the lattice $\mathbb{Z}$, with at most one particle per site (exclusion principle). Each particle has an independent exponential clock and jumps to the right by one when it rings, provided the neighbouring site is empty. The q-TASEP is a generalization introduced in [BC14]. In this model, the i-th particle jumps to the right by one at rate $a_i(1 - q^{\mathrm{gap}})$ where the gap is the number of consecutive empty sites to the next particle on the right. The parameter $q \in (0, 1)$ can hence be seen as a repulsion strength between particles.

Among many other stochastic models, the TASEP and the q-TASEP lie in the KPZ universality class, named from the Kardar-Parisi-Zhang stochastic partial differential equation modelling the growth of interfaces. The most prominent common features of models belonging to this class are fluctuations on a scale $t^{1/3}$, spatial correlations on a scale $t^{2/3}$, and Tracy-Widom type statistics (see the review [Cor12] on the KPZ class).

Borodin-Corwin's theory of Macdonald processes [BC14] provides an algebraic framework to study integrable models in the KPZ universality class, extending the Schur processes [Oko01, OR03] which prove useful in the study of TASEP and other models. Macdonald processes are a two parameter family of measures on interlacing sequences of integer partitions, or Gelfand-Tsetlin patterns. The probability of a given pattern is expressed as a product of Macdonald functions, which are symmetric functions depending on two formal parameters q and t. Various particular choices of the parameters are examined in [BC14], leading to applications to several stochastic models from statistical mechanics. When the parameter t is set to zero, Macdonald functions degenerate to q-Whittaker functions. Besides the potential interest of the model itself, the introduction of q-TASEP is natural since it is a marginal of the q-Whittaker process in the same way as the TASEP is a marginal of the Schur process. When q tends to 1, the q-Whittaker functions become Whittaker functions whose connections with directed polymers were established in [OC12]. The limit of the q-Whittaker process when q goes to 1 is investigated in [BC14, BCF14]. Hence, q-TASEP interpolates between TASEP when $q = 0$, and the O'Connell-Yor semi-discrete polymer when $q \to 1$, under a particular scaling of the parameters [BCF14]. After further rescaling the space, the semi-discrete polymer model converges to the continuum directed random polymer. The predictions of the KPZ universality class about the scaling exponents and the limit theorem towards the Tracy-Widom distribution are proved in [Joh00] for TASEP and in [BCF14] for the one-point distribution of the free energy of continuous polymers. As for the q-TASEP, when all particles have the same speed $a_i \equiv 1$ and starting from the so-called step initial condition, Ferrari and Vető [FV13] show that the properly rescaled position of particles converges in law to the emblematic GUE Tracy-Widom distribution. Moreover, they confirm the KPZ scaling theory [Spo12, KMHH92], which conjecturally predicts the exact value of all non-universal constants arising in the law of large numbers and the Tracy-Widom limit theorem.

Another ubiquitous probability distribution appearing in the KPZ universality class is a generalization of the Tracy-Widom distribution called the BBP distribution. It first arose in spiked random matrix theory [BBAP05]. A phase transition happens for perturbed Gaussian ensembles of hermitian matrices, and the fluctuations at the edge of the spectrum lie in the Gaussian regime or in the Tracy-Widom regime, according to the rank and the structure of the perturbation. Using connections between sample covariance matrices and last passage percolation (LPP), Baik, Ben Arous and Péché [BBAP05] explain how their results translate in terms of last passage percolation on the first quadrant where the first finitely many rows have different means. Using then a coupling between LPP and TASEP, Baik [Bai06] explains that one also observes the BBP transition for the fluctuations of the current in TASEP started from step initial condition, where all but finitely many particles have rate 1 and a few have a smaller rate. The number of slower particles plays the same role as the rank of the perturbation in the matrix model. On the other degeneration, that is when q goes to 1, Borodin, Corwin and Ferrari [BCF14] show the same phase transition for the fluctuations of the free energy of the O'Connell-Yor directed polymer when adding a local perturbation of the environment in a quite similar way.

A finite number of slower particles in the q-TASEP creates a shock on a macroscopic scale. Our purpose is to show the same phase transition for the fluctuations of particles around their hydrodynamic limit, depending on the minimum rate and the number of slower particles. We adapt the asymptotic analysis made in [FV13]. The main technical difference is the following: in [FV13], the fluctuation result is proved with a slightly restrictive condition on the macroscopic position of particles, which forbids to study the very first particles. It concerns a $\mathcal{O}$(time) quantity of particles though. In our work, we do not assume this condition to hold, confirming that it was purely technical as suspected by Ferrari and Vető.

2.1.1 The q-TASEP

The q-TASEP is a continuous-time Markov process described by the coordinates of particles $X_N(t) \in \mathbb{Z}, N \in \mathbb{Z}_{>0}, t \in \mathbb{R}_+$. Its infinitesimal generator is given by

$$(Lf)(\mathbf{x}) = \sum_{k \in \mathbb{N}^*} a_k (1 - q^{x_{k-1} - x_k - 1})(f(\mathbf{x}^k) - f(\mathbf{x})) \tag{2.1}$$

where $\mathbf{x} = (x_0, x_1, \dots)$ is such that $x_i > x_{i+1}$ for all i, $\mathbf{x}^k$ is the configuration where x_k is increased by one, and by convention $x_0 = +\infty$. The step initial condition corresponds to $\forall i \in \mathbb{N}^*, x_i(0) = -i$.

Let us first focus on the case where all particles have equal hopping rates, $a_i \equiv 1$. In this case, the gaps between particles $(x_i - x_{i+1} - 1)_i$ have the same dynamics as a q-TAZRP (q-deformed Totally Asymmetric Zero-Range Process) introduced in [SW98a], for which general results on invariant distributions of zero-range processes apply [And82]. Hence, [BC14] shows that translation invariant extremal invariant measures are renewal processes μ_r for $r \in [0, 1)$ on $\mathbb{Z}$, with renewal measure according to the q-Geometric distribution of parameter r, i.e.

$$\mu_r(x_i - x_{i+1} - 1 = k) = (r, q)_\infty \frac{r^k}{(q, q)_k}$$

where $(z, q)_k = (1 - z)(1 - qz) \dots (1 - q^{k-1} z)$.

One expects that the rescaled particle density $\rho(x, \tau)$, given heuristically by

$$\rho(x, \tau) = \lim_{t \to \infty} \mathbb{P}(\text{ There is a particle at } \lfloor xt \rfloor \text{ at time } t\tau),$$

satisfies the PDE

$$\frac{\partial}{\partial \tau} \rho(x, \tau) + \frac{\partial}{\partial x} j(\rho)(x, \tau) = 0 \tag{2.2}$$

where $j(\rho)$ is the particle current at density ρ.

By local stationarity assumption, we mean that gaps between particles are locally distributed according to i.i.d. q-geometric random variables for some parameter r which depends on the macroscopic position. Under this assumption and using the particle conservation PDE (2.2), one can guess the deterministic profile of the q-TASEP, see [FV13, Section 3] for details. In order to state this result, we need some preliminary definitions. We fix the parameter $q \in (0, 1)$ and choose a real number $\theta > 0$ which parametrizes the macroscopic position of particles, as explained below.

Definition 2.1.1 ([FV13])**.** We recall the definition of the q-Gamma function

$$\Gamma_q(z) = (1 - q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty}.$$

Then the q-digamma function is defined by

$$\Psi_q(z) = \frac{\partial}{\partial z} \log \Gamma_q(z).$$

For $q \in (0,1)$ and $\theta > 0$, we also define the functions

$$\kappa \equiv \kappa(q,\theta) = \frac{\Psi_q'(\theta)}{(\log q)^2 q^\theta}, \tag{2.3}$$

$$f \equiv f(q,\theta) = \frac{\Psi_q'(\theta)}{(\log q)^2} - \frac{\Psi_q(\theta)}{\log q} - \frac{\log(1-q)}{\log q}, \tag{2.4}$$

$$\chi \equiv \chi(q,\theta) = \frac{\Psi_q'(\theta)\log q - \Psi_q''(\theta)}{2}. \tag{2.5}$$

Then, the following law of large numbers holds when N goes to infinity,

$$\frac{X_N(\tau = \kappa N)}{N} \longrightarrow f - 1.$$

Let us explain the arguments leading to this result. Under local stationarity assumption, for a real number x such that around xt, gaps between particles are distributed as q-Geometric random variables of parameter r, we have

$$\rho(x,t) = \frac{1}{1+\mathbb{E}[\text{gap}]} = \frac{\log q}{\log(q) + \log(1-q) + \Psi_q(\log_q r)}. \tag{2.6}$$

Writing $x = x(r)$ and after the change of variables $\log_q r = \theta$, the PDE (2.2) implies that $x(\theta) = (f(q,\theta) - 1)/\kappa(q,\theta)$. Finally, integrating this density gives a law of large numbers for the integrated current of particles, or the equivalent statement on $X_N(\tau)$ given above. Moreover, under local stationarity assumption, the gaps between consecutive particles around particle N at time $\kappa(q,\theta)N$ are distributed according to i.i.d. q-Geometric random variables of parameter q^θ. The averaged hopping rate (or the averaged speed of a tagged particle) is then $\mathbb{E}[1 - q^{\text{gap}}] = q^\theta$.

Remark 2.1.2. One could choose the time τ to be the free parameter which tends to infinity, but the formulae are slightly simpler when N tends to infinity and τ depends on N.

2.1.2 Main result

In this paper, we aim to study a q-TASEP started form step initial condition where all but finitely many particles have rate 1, and some particles are slower. Notice that as we will see in the proofs, the case of a finite number of faster particles does not change anything to the macroscopic behaviour.

Let $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ be the rates of the slower particles, and α be the rate of the slowest particle and suppose that $k \leqslant m$ particles have rate α. For later use, it is convenient to set the notation $A := \log_q(\alpha)$.

The slower particles create a shock on a macroscopic scale and influence the law of large numbers. The particles which are concerned by the shock are all the particles that, in absence of slower particles, would have an averaged hopping rate greater than α. In order to state the results precisely, we need to define some additional functions.

Definition 2.1.3. For $q \in (0,1)$ and $\theta > 0$, we define

$$g \equiv g(q,\theta) = \frac{\Psi_q'(\theta)}{(\log q)^2}\frac{\alpha}{q^\theta} - \frac{\Psi_q(A)}{\log q} - \frac{\log(1-q)}{\log q}, \tag{2.7}$$

$$\sigma \equiv \sigma(q,\theta) = \Psi_q'(\theta)\frac{\alpha}{q^\theta} - \Psi_q'(A). \tag{2.8}$$

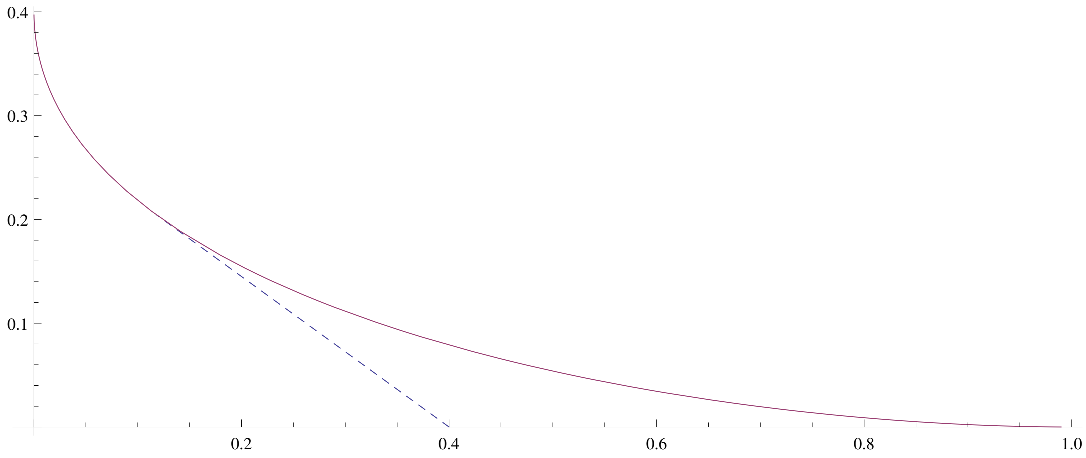

Figure 2.1: Limit shapes of $\frac{1}{\tau}(X_N(\tau)+N, N)$ for $q=0.6$. The solid line corresponds to $\alpha=1$ (no slow particle). The dashed line corresponds to $\alpha=0.4$ (one or several slower particles). Note that the curved line is the parametric curve $(f/\kappa, 1/\kappa)$, whereas the straight line part is the parametric curve $(g/\kappa, 1/\kappa)$, when θ ranges from 0 to $\log_q \alpha$, i.e. θ satisfying the condition $\alpha < q^\theta$.

Then the following law of large numbers holds, when N goes to infinity,

$$\frac{X_N(\tau=\kappa N)}{N} \longrightarrow \begin{cases} f-1 & \text{when} \quad \alpha > q^\theta, \\ g-1 & \text{when} \quad \alpha \leqslant q^\theta. \end{cases}$$

The limit shape of $\frac{1}{\tau}(X_N(\tau)+N, N)$ is drawn in Figure 2.1. One sees that the limiting profile is linear when $\alpha < q^\theta$. It means that the density of particles is constant inside the shock and particles have an averaged speed α.

The scaling theory of single-step growth models explained in [Spo12] and the KPZ class conjecture give a way to compute the non universal constants arising in the law of large numbers, and the scale and precise variance of fluctuations. Nevertheless, these results are expected to hold only at points where the limiting shape is strictly convex. The results in [FV13] confirm the conjecture for a q-TASEP where all particles have equal hopping rates. In the presence of slower particles, the limiting shape is linear inside the shock (cf Figure 2.1), and the fluctuations are not predicted by the KPZ class conjecture.

Remark 2.1.4. In [Spo12], the convexity condition is given on the limit profile of the height function, but this is equivalent.

Let us explain the scalings that we use in the paper. When θ is such that $q^\theta < \alpha$, particles around the macroscopic position parametrized by θ have speed q^θ and are asymptotically independent from the slower particles. We expect to observe Tracy-Widom fluctuations on a $N^{1/3}$ scale with spatial correlation on a $N^{2/3}$ scale. Hence, the time $\tau(N,c)$ is set as

$$\tau(N,c) = \kappa N + c q^{-\theta} N^{2/3},$$

where c is a free but fixed parameter. The macroscopic position of the N^{th} particle, denoted $p(N,c)$, is given by

$$p(N,c) = (f-1)N + cN^{2/3} - c^2 \frac{(\log q)^3}{4\chi} N^{1/3} \ .$$

When θ is such that $\alpha < q^{\theta}$, we expect the fluctuations to live on the $N^{1/2}$ scale, although the limiting law is not necessarily Gaussian. We set

$$\tau^*(N,c) = \kappa N + cN^{1/2}/\alpha,$$

and the macroscopic position is

$$p^*(N,c) = (g-1)N + cN^{1/2}.$$

The aim is to study the fluctuations of $X_N(\tau(N,c))$ (resp. $X_N(\tau^*(N,c))$) around $p(N,c)$ (resp. $p^*(N,c)$).

Next, we define classical probability distributions from random matrix theory in a convenient way for our purposes.

Definition 2.1.5 (Distribution functions)**.**

1. The distribution function $F_{\mathrm{GUE}}(x)$ of the GUE Tracy-Widom distribution is defined by $F_{\mathrm{GUE}}(x) = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)}$ where K_{Ai} is the Airy kernel,

$$K_{\mathrm{Ai}}(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{-2i\pi/3}\infty}^{e^{2i\pi/3}\infty} \mathrm{d}w \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \mathrm{d}z \frac{e^{z^3/3-zu}}{e^{w^3/3-wv}} \frac{1}{z-w},$$

where the contours for z and w do not intersect.

2. Let $\mathbf{b} = (b_1,\dots,b_k) \in \mathbb{R}^k$. The BBP distribution of rank k from [BBAP05] is defined by $F_{\mathrm{BBP,k,}\mathbf{b}}(x) = \det(I - K_{\mathrm{BBP,k,}\mathbf{b}})_{\mathbb{L}^2(x,+\infty)}$ where $K_{\mathrm{BBP,k,}\mathbf{b}}$ is given by

$$K_{\mathrm{BBP,k,}\mathbf{b}}(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{-2i\pi/3}\infty}^{e^{2i\pi/3}\infty} \mathrm{d}w \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \mathrm{d}z \frac{e^{z^3/3-zu}}{e^{w^3/3-wv}} \frac{1}{z-w}\left(\frac{z-b}{w-b}\right)^k,$$

where the contour for w passes to the right of the b_i's, and the contours for z and w do not intersect.

3. $G_k(x)$ is the distribution of the largest eigenvalue of a $k\times k$ GUE, which also has a Fredholm determinant representation. $G_k(x) = \det(I - H_k)_{\mathbb{L}^2(x,+\infty)}$, where H_k is the Hermite kernel given by

$$H_k(u,v) = \frac{c_{k-1}}{c_k}\frac{p_k(u)p_{k-1}(v) - p_{k-1}(u)p_k(v)}{u-v} e^{-(u^2+v^2)/4},$$

where $c_n = 1/((2\pi)^{1/4}\sqrt{n!})$ and $(p_n)_{n\geqslant 0}$ is a family of orthogonal polynomials determined by $\int_{-\infty}^{\infty} p_m(t)p_n(t)e^{-t^2/2}\mathrm{d}t = \delta_{mn}$. The kernel H_k has an integral representation

$$H_k(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{i(\varphi-\pi)}\infty}^{e^{i(\pi-\varphi)}\infty} \mathrm{d}w \int_{e^{i(\gamma-\pi/2)}\infty}^{e^{i(\pi/2-\gamma)}\infty} \mathrm{d}z \frac{e^{z^2/2-zu}}{e^{w^2/2-wv}} \frac{1}{z-w}\left(\frac{z}{w}\right)^k,$$

with $\varphi,\gamma \in (0,\pi/4)$, and where the contour for w passes to the right of 0, and the contours do not intersect. For the equivalence between these formulas, see e.g. [BK05] and references therein.

We are now able to state our main result.

Theorem 2.1.6. *Let k be the number of particles having rate α.*

- *If $q^\theta < \alpha$, then writing $X_N(\tau(N)) = (f-1)N + cN^{2/3} - c^2\frac{(\log q)^3}{4\chi}N^{1/3} + \frac{\chi^{1/3}}{\log q}\xi_N N^{1/3}$, we have for any $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{GUE}}(x).$$

- *If $q^\theta = \alpha$ then writing again $X_N(\tau(N)) = (f-1)N + cN^{2/3} - c^2\frac{(\log q)^3}{4\chi}N^{1/3} + \frac{\chi^{1/3}}{\log q}\xi_N N^{1/3}$, we have for any $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{BBP,k,}\mathbf{b}}(x)$$

where $\mathbf{b} = (b, \dots, b)$ with $b = \frac{c(\log q)^2}{2\chi^{2/3}}$.

- *If $q^\theta > \alpha$, then writing $X_N(\tau^*(N)) = (g-1)N + cN^{1/2} + \frac{\sigma^{1/2}}{\log q}\xi_N N^{1/3}$, we have for any $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = G_k(x).$$

Remark 2.1.7. These results on the asymptotic positions of particles readily translate in terms of current of particles or height function, as in [FV13, Theorem 2.9].

Remark 2.1.8. Furthermore, it is possible to observe the BBP distribution $F_{\mathrm{BBP},k,\mathbf{b}}$ for any arbitrary vector $\mathbf{b}$. One has to perturb the rates on a scale $N^{-1/3}$. Let fix some $\theta > 0$ and assume that for $1 \leqslant i \leqslant k$, $a_i = q^{\theta + \tilde{b}_i N^{-1/3}}$ and the rates of all other particles are higher than q^θ. Then, if the random variable ξ_N is defined as in Theorem 2.1.6, $\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{BBP,k,}\mathbf{b}}(x)$ where $\mathbf{b} = (b + \tilde{b}_1, \dots, b + \tilde{b}_1)$ with $b = \frac{c(\log q)^2}{2\chi^{2/3}}$ as before. In terms of Macdonald processes, this perturbation of the rates corresponds to a perturbation of the parameters of the q-Whittaker process. It is the precise analogue of the perturbation of the parameters of the Whittaker process applied to the O'Connell-Yor semi-discrete random polymer, for which the same result holds for the fluctuations of the free energy [BCF14, Theorem 2.1].

2.2 Asymptotic analysis

In this section, we prove the main theorem 2.1.6. The asymptotic analysis we present here is an instance of Laplace's method closely adapted from [FV13]. Fix a $\theta > 0$ and the parameter $c \in \mathbb{R}$. We start from a Fredholm determinant representation for the q-Laplace transform of $q^{X_N(\tau)}$, which characterizes the law of $X_N(\tau)$. It was first proved in [BC14, BCS14] in a slightly different form, and in [BCF14] in the form which seems the most convenient for an asymptotic analysis. Before stating this result, we need to define some integration contours in the complex plane.

Definition 2.2.1 (see Figure 2.2)**.** We define a family of contours $\tilde{\mathcal{C}}_{\bar\alpha,\varphi}$ for $\bar\alpha > 0$ and $\varphi \in (0, \pi/2)$ by

$$\tilde{\mathcal{C}}_{\bar\alpha,\varphi} = \{\bar\alpha + e^{i\varphi \operatorname{sgn}(y)}|y|, y \in \mathbb{R}\},$$

oriented from bottom to top. For every $w \in \tilde{\mathcal{C}}_{\bar\alpha,\varphi}$, we define a contour $\tilde{\mathcal{D}}_w$ by

$$\tilde{\mathcal{D}}_w = \,]R - i\infty, R - id] \cup]R - id, 1/2 - id] \cup]1/2 - id, 1/2 + id] \\ \cup]1/2 + id, R + id] \cup]R + id, R + i\infty[$$

oriented from bottom to top, where $R, d > 0$ are chosen such that:

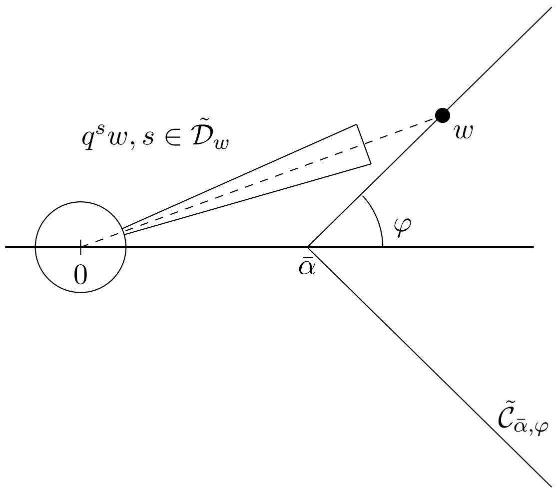

Figure 2.2: The contours in Definition 2.2.1. One sees that $q^s w, s \in \tilde{\mathcal{D}}_w$ stays to the left of $\tilde{\mathcal{C}}_{\bar{\alpha},\varphi}$.

(i) Let $b = \pi/4 - \varphi/2$, then $\arg(w(q^s - 1)) \in (\pi/2 + b, 3\pi/2 - b)$ for any $s \in \tilde{\mathcal{D}}_w$.
(ii) $q^s w, s \in \tilde{\mathcal{D}}_w$ stays to the left of $\tilde{\mathcal{C}}_{\bar{\alpha},\varphi}$.

These contours always exist (see Remark 4.9 in [BCF14]): to check condition (ii), it is enough to prove that the points $wq^{1/2\pm ir}$ stay on the left of $\tilde{\mathcal{C}}_{\bar{\alpha},\varphi}$ which follows from simple geometric arguments for d small enough. It can be seen in Figure 2.2. To satisfy condition (i), the argument of $wq^s - w$ can be made as small as we want choosing d small enough and R large enough.

Theorem 2.2.2. *Let $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ and $N \geqslant \max_{1\leqslant j\leqslant m}\{i_j\}$. Then,*

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_N(t)+N}; q)_\infty}\right] = \det(I + \tilde{K}_\zeta)_{\mathbb{L}^2(\tilde{\mathcal{C}}_{\bar{\alpha},\varphi})} \tag{2.9}$$

for any $\varphi \in (0, \pi/4)$ provided the condition $0 < \bar{\alpha} < \alpha$ is satisfied. The operator $\tilde{K}_\zeta$ is defined by its integral kernel

$$\tilde{K}_\zeta(w, w') = \frac{1}{2i\pi}\int_{\tilde{\mathcal{D}}_w} \mathrm{d}s\Gamma(-s)\Gamma(1+s)\,(-\zeta)^s\, g_{w,w'}(q^s),$$

where

$$g_{w,w'}(q^s) = \frac{\exp(tw(q^s-1))}{q^s w - w'}\left(\frac{(q^s w; q)_\infty}{(w;q)_\infty}\right)^{N-m}\prod_{1\leqslant j\leqslant m}\left(\frac{(q^s w/a_{i_j}; q)_\infty}{(w/a_{i_j}; q)_\infty}\right).$$

Remark 2.2.3. The condition $0 < \bar{\alpha} < \alpha$ ensures that all the poles for the variable w in $g_{w,w'}(q^s)$ are inside the contour $\tilde{\mathcal{C}}_{\bar{\alpha},\varphi}$.

Proof. Let us explain how this theorem is a rephrasing of a known result on Macdonald processes. The (ascending) Macdonald processes introduced in [BC14] are a family of probability measures on sequences of integer partitions $\lambda^1, \lambda^2, \dots, \lambda^N$, where λ^i has at most i non-zero components, and the sequence satisfies the interlacing condition $\lambda_{j+1}^k \leqslant \lambda_j^{k-1} \leqslant \lambda_j^k$, for all $1 \leqslant j \leqslant k-1 < N$. The Macdonald measures are a family of measures on integer partitions such that the marginals of Macdonald processes are Macdonald measures cf [BC14, paragraph 2.2.2].

The probability of a given configuration is expressed as a product of Macdonald functions, which are symmetric functions in infinitely many variables, such that the coefficient of each monomial is a rational function in two parameters q and t. In order to build a genuine positive measure, one has to properly specialize Macdonald functions and consider q and t as real parameters.

Different choices for the parameters q and t are examined in [BC14]. When $t = 0$, the study of the dynamics preserving Macdonald processes leads to the definition of the q-TASEP. Indeed, under a specialization of the Macdonald process with parameters $(a_1, \dots, a_N; \rho_\tau)$ where ρ_τ is the Plancherel specialization of parameter τ, the marginal $(\lambda_k^k)_{1\leqslant k\leqslant N}$ has Markovian dynamics which is exactly that of the q-TASEP. More precisely, $(X_k(\tau) + k)_{1\leqslant k\leqslant N}$ has the same distribution as the marginal $(\lambda_k^k)_{1\leqslant k\leqslant N}$ for any time τ. The theorem is now a reformulation of Theorem 4.13 in [BCF14] which proves the same formula for $\mathbb{E}\left[1/(\zeta q^{\lambda_N^N}; q)_\infty\right]$ when λ^N is distributed according to the Macdonald measure under specialization $(a_1, \dots, a_N; \rho_\tau)$.

□

We make the change of variables:

$$w = q^W, \quad w' = q^W, \quad s + W = Z.$$

This demands to introduce new integration contours for the variables Z, W and W', depicted in Figure 2.3. Let $\bar{A} = \log_q(\alpha)$ and let $\mathcal{C}_{\bar{A},\varphi}$ be the image of $\tilde{\mathcal{C}}_{\bar{\alpha},\varphi}$ under the map $x \mapsto \log_q x$. For every $W \in \mathcal{C}_{\bar{A},\varphi}$, let $\mathcal{D}_W = \{\bar{A} + \sigma + i\mathbb{R}\} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_{k_W}$, the value of $\sigma > 0$ to be chosen later [1], where $\mathcal{E}_1, \dots, \mathcal{E}_{k_W}$ are small circles around the residues coming from the sine at $W+1, W+2, \dots, W+k_W$.

More precisely, the vertical line is modified in a neighbourhood of size δ around the real axis as in Figure 2.3, and we choose σ such that the poles coming from the sine inverse are at a distance from the vertical line at least $\sigma/2$. To make this possible, the vertical lines of the contour are chosen to have real part $\bar{A} + \sigma$ or $\bar{A} + 2\sigma$.

We obtain, as in [FV13], the kernel

$$\hat{K}_\zeta(W, W') = \frac{q^W \log q}{2i\pi} \int_{\mathcal{D}_W} \frac{\mathrm{d}Z}{q^Z - q^{W'}} \frac{\pi}{\sin(\pi(W - Z))} \\ \frac{(-\zeta)^Z \exp(\tau q^Z + (N-m)\log(q^Z; q)_\infty + \sum_{j=1}^m \log(q^Z/a_{i_j}; q)_\infty)}{(-\zeta)^W \exp(\tau q^W + (N-m)\log(q^W; q)_\infty) + \sum_{j=1}^m \log(q^W/a_{i_j}; q)_\infty)}.$$

2.2.1 Case $\alpha > q^\theta$, Tracy-Widom fluctuations.

Fix $\theta > 0$ such that $\alpha > q^\theta$ and $c \in \mathbb{R}$. We want to study the limit of

$$\mathbb{P}\left(\frac{X_N(\tau(N,c)) - p(N,c)}{\chi^{1/3}/(\log q)N^{1/3}} < x\right). \tag{2.10}$$

1. Note that the real number σ that we use here has nothing to do with the standard deviation σ in Equation 2.8, but we allow this abuse of notations to keep the same notations as in [FV13]

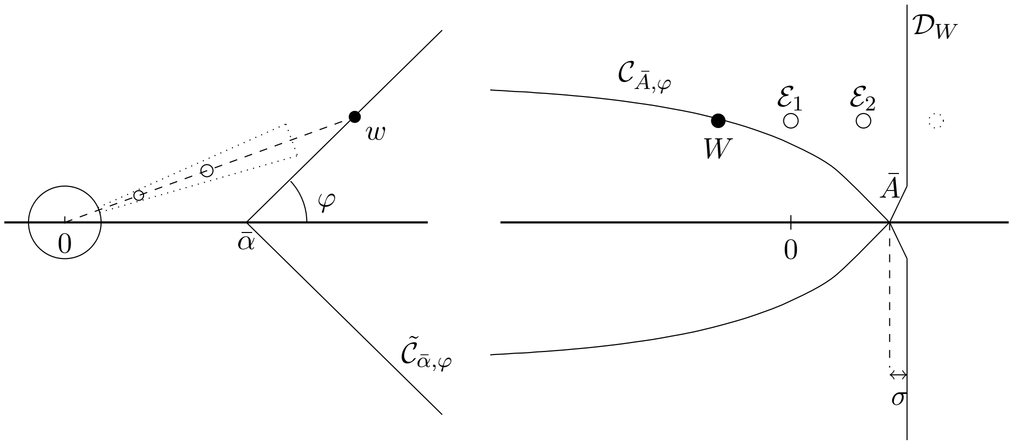

Figure 2.3: The contours for the variables Z, W and W' are on the right-hand-side. The left-hand-side is the image by the map $Z \mapsto q^Z$. In this example, $k_W = 2$.

The function $z \mapsto 1/(z;q)_\infty$ converges uniformly as an infinite product for $z \in]-\infty, 0]$. Thus when z goes to $-\infty$, then $1/(z;q)_\infty$ goes to zero, and when z goes to zero then $1/(z;q)_\infty$ goes to 1. Modulo a justification of the exchange between expectation and limit that we explicit in the end of this subsection, if

$$\zeta = -q^{-p(N,c)-N-\frac{\chi^{1/3}}{\log q}xN^{1/3}}$$

then (2.10) and $\mathbb{E}\left[1/(\zeta q^{X_N(\tau)};q)_\infty\right]$ have the same limit. Thus, in the following of this subsection, we set ζ as above. We fix also $\bar{\alpha} = q^\theta$ (or equivalently $\bar{A} = \theta$). As we assume $\alpha > q^\theta$, the condition $0 < \bar{\alpha} < \alpha$ in Theorem 2.2.2 is satisfied. We then obtain $\det(I + \tilde{K}_\zeta)_{\mathbb{L}^2(\tilde{\mathcal{C}}_{\bar{\alpha},\varphi})} = \det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\varphi})}$ where

$$K_x(W, W') = \\ \frac{q^W \log q}{2i\pi} \int_{\mathcal{D}_W} \frac{\mathrm{d}Z}{q^Z - q^{W'}} \frac{\pi}{\sin(\pi(Z-W))} \frac{\exp(Nf_0(Z) + N^{2/3}f_1(Z) + N^{1/3}f_2(Z))}{\exp(Nf_0(W) + N^{2/3}f_1(W) + N^{1/3}f_2(W))} \frac{\phi(Z)}{\phi(W)}, \tag{2.11}$$

with

$$\begin{aligned} f_0(Z) &= -f(\log q)Z + \kappa q^Z + \log(q^Z;q)_\infty, \\ f_1(Z) &= -c(\log q)Z + cq^{Z-\theta}, \\ f_2(Z) &= c^2\frac{(\log q)^4}{4\chi}Z - \chi^{1/3}xZ, \\ \phi(Z) &= \frac{\prod_{j=1}^m (q^Z/a_{i_j};q)_\infty}{((q^Z;q)_\infty)^m}. \end{aligned}$$

Let us describe the idea of Laplace's method in our context. The asymptotic behaviour of the kernel is governed by the variations of the real part of f_0. In the sequel, we exhibit steep-descent contours, which allows us to prove that in the large N limit, the main contribution to the Fredholm determinant is localized in a neighbourhood of θ which is the critical point of $\mathfrak{Re}[f_0]$. Then, using estimates and Taylor expansions for the argument of the exponential inside the kernel, we prove the limit. Due to the difficulty to simultaneously find a steep-descent path for the contour of the Fredholm determinant and to control the extra residues coming from the sine inverse in formula (2.11), the authors in [FV13] impose a technical condition $q^\theta \leqslant \frac{1}{2}$, suspecting that it was purely technical (see Remark 2.5). In order to get rid of this condition, we do not choose exactly the same contours. Our contour for the variable W is $\mathcal{C}_{\theta,\pi/4}$ instead of $\mathcal{C}_{\theta,\varphi}$ for φ close to $\pi/2$. Indeed, for $\varphi \neq \pi/4$, the contour $\mathcal{C}_{\theta,\varphi}$ is not necessarily steep-descent for $-\mathfrak{Re}[f_0]$ when $q^\theta > 1/2$.

For later use, we give two useful series representations for Ψ_q and its derivative

$$\Psi_q(Z) = -\log(1-q) + \log q \sum_{k=0}^{\infty} \frac{q^{Z+k}}{1-q^{Z+k}}, \tag{2.12}$$

$$\Psi_q'(Z) = (\log q)^2 \sum_{k=0}^{\infty} \frac{q^{Z+k}}{(1-q^{Z+k})^2}. \tag{2.13}$$

In general, we parametrize the contour $\mathcal{C}_{\bar{A},\varphi}$ by $W(s) = \log_q(\alpha + |s|e^{i\varphi \operatorname{sgn}(s)})$ for $s \in \mathbb{R}$. For instance, the contour $\mathcal{C}_{\theta,\pi/4}$ is parametrized by $W(s) = \log_q(q^\theta + |s|e^{\operatorname{sgn}(s)i\pi/4})$.

Lemma 2.2.4 ([FV13]). *1) The two following expressions for f_0' are useful:*

$$\begin{aligned} f_0'(Z) &= \frac{\Psi_q'(\theta)}{\log q}(q^{Z-\theta} - 1) + \Psi_q(\theta) - \Psi_q(Z) & (2.14)\\ &= -\log q \sum_{k=0}^{\infty} \frac{q^{2k}(q^\theta - q^Z)^2}{(1-q^{\theta+k})^2(1-q^{Z+k})}. & (2.15) \end{aligned}$$

2) We have that

$$\frac{\mathrm{d}}{\mathrm{d}Y}\left(\mathfrak{Re}\left[f_0(X+iY)\right]\right) = -\sin(Y\log q)\log q \sum_{k=0}^{\infty} q^{X+k}\left(\frac{1}{(1-q^{\theta+k})^2} - \frac{1}{|1-q^{X+iY+k}|^2}\right). \tag{2.16}$$

3) The contour $\mathcal{C}_{\theta,\pi/4}$ is steep-descent for $-\mathfrak{Re}\left[f_0\right]$, in the sense that the function $s \mapsto \mathfrak{Re}\left[f_0(W(s))\right]$ is increasing for $s \geqslant 0$, where $W(s)$ is a parametrization of the contour $\mathcal{C}_{\theta,\pi/4}$.

4) The function $\mathfrak{Re}\left[f_0\right]$ is periodic on $\{\theta + \sigma + i\mathbb{R}\}$ with period $2\pi/|\log q|$. Moreover, $t \mapsto \mathfrak{Re}\left[f_0(\theta+\sigma+it)\right]$ is decreasing on $[0, -\pi/\log q]$ and increasing on $[\pi/\log q, 0]$, for any $\sigma > 0$.

Proof. Equations (2.15) and (2.14) correspond to Equations (6.19) and (6.22) in [FV13]. Equation (2.16) is Equation (6.24) in [FV13] with $X = \theta + \gamma$ and $Y = t$. 4) follows directly from this expression. 3) is a particular case of of Lemma 6.8 in [FV13] and still holds when $q^\theta > 1/2$. Indeed, after some algebra,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{Re}\left[f_0(W(s))\right]\right) = \sum_{k=0}^{\infty} \frac{q^{2k}s^2\sqrt{2}/2\left(q^k s^2 + q^\theta(1-q^{\theta+k})\right)}{(1-q^{\theta+k})^2|1-q^{\theta+k} - e^{i\pi/4}sq^k|^2|q^{\theta+e^{i\pi/4}s}|^2} > 0.$$

$\square$

Lemma 2.2.5 ([FV13])**.** *The kernel $K_x(W(s),W')$ has exponential decay, in the sense that there exist N_0, $s_0 \geqslant 0$ and $c > 0$ such that for all $s > s_0$ and $N > N_0$,*

$$|K_x(W(s),W')| \leqslant \exp(-cNs).$$

Proof. This is a particular case of Lemma 6.10 in [FV13], i.e. when $\varphi = \pi/4$. The proof consists in estimating separately the contributions of the vertical line $\theta + \sigma + i\mathbb{R}$ and the small circles $\mathcal{E}_1, \dots, \mathcal{E}_{k_W}$. The factor $\exp(N(f_0(Z) - f_0(W)))$ inside the kernel commands the asymptotic behaviour. Thus the result boils down to showing that there exists a constant $c > 0$ such that for $N > N_0$, $s > s_0$ and any $Z \in \mathcal{D}_W$,

$$\mathfrak{Re}\left[f_0(Z) - f_0(W(s))\right] < -cs.$$

This follows from the properties of the function f_0 given in Lemma 2.2.4. Note that this result is also a degeneration of Lemma 2.2.13 proved thereafter. □

The previous Lemma allows to extend the contour $\mathcal{C}_{\theta,\varphi}$ with $\varphi \in (0, \pi/4)$ in Theorem 2.2.2 to $\varphi = \pi/4$, without altering the Fredholm determinant $\det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\varphi})}$. Indeed, for $\varphi \in (0, \pi/4)$, it is known from the proof of Theorem 4.13 in [BCF14] that the kernel decays exponentially on $\mathcal{C}_{\theta,\varphi}$. For large N, Lemma 2.2.5 gives an exponential bound on the kernel K_x along the tails of the contour $\mathcal{C}_{\theta,\pi/4}$, i.e. for $|s| \geqslant s_0$. The behaviour of the kernel around $s = 0$ is logarithmic, so that for a fixed $N > N_0$ one has (cf also [FV13, equation 6.28])

$$|K_x(W(s),W(s'))| \leqslant C\exp(-cN|s|) + C(\log|s|)_-(\log|s'|)_-. \tag{2.17}$$

where $(x)_-$ denotes the negative part of x. Hence, for any fixed $N > N_0$, each term in the series expansion of the Fredholm determinant is constant when φ varies in $(0, \pi/4]$, yielding the validity of the contour deformation for the Fredholm determinant.

Next, we want to show that the parts of the contours which give the main contribution to the Fredholm determinant are in a neighbourhood of θ. When $q^\theta \leqslant 1/2$ and the contour for the variables W and W' is $\mathcal{C}_{\theta,\varphi}$ with φ close to $\pi/2$, it is proved in Proposition 6.3 of [FV13]. In order to get rid of the condition $q^\theta \leqslant 1/2$, we need to control the real part of f_0 on the small circles $\mathcal{E}_1, \dots, \mathcal{E}_{k_W}$. This is done by the following Lemma.

Lemma 2.2.6. *There exists $\eta > 0$ such that for any $W \in \mathcal{C}_{\theta,\pi/4}$, $\mathfrak{Re}(f_0(W) - f_0(W+j)) > \eta$, for all $j = 1, \dots, k_W$.*

Proof. It is proved in Lemma 6.10 in [FV13] (in the proof thereof, more exactly) that for W far enough from θ, i.e. for $W = W(s)$ with $|s| > s_0$,

$$\mathfrak{Re}\left[f_0(W(s)) - f_0(W(s)+j)\right] > c \cdot |s|$$

for some $c > 0$. Thus, we can consider the residues lying only in a compact domain, and we are left to prove that $\mathfrak{Re}\left[f_0(W) - f_0(W+j)\right] > 0$ for each residue.

We split the proof into two cases according to the sign of $\mathfrak{Re}\left[q^{W+j} - q^\theta\right]$, or in other words, according to the relative position of the residue $W + j$ and the contour $\mathcal{C}_{\theta,\pi/2}$. By symmetry, we can consider only the residues above the real axis.

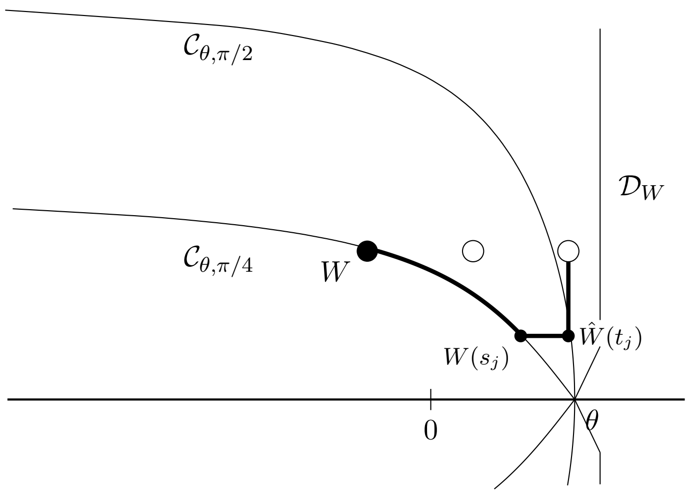

Figure 2.4: The thick line is the path from W to $W+j$ in Case 2, for $j=2$.

Case 1 : $\mathfrak{Re}\left[q^{W+j}-q^{\theta}\right]\geqslant 0$. This condition geometrically means that $W+j$ lies on the left of $\mathcal{C}_{\theta,\pi/2}$, i.e. between $\mathcal{C}_{\theta,\pi/4}$ and $\mathcal{C}_{\theta,\pi/2}$. We show that on the straight line from W to $W+j$, $\mathfrak{Re}[f_0]$ is decreasing. For that purpose, it is enough to prove that $\mathfrak{Re}\left[f_0'(W+X)\right]<0$ for $X\in(0,j)$. From the expression of f_0' in Lemma 2.2.4 eq. (2.15),

$$\mathfrak{Re}\left[\frac{\mathrm{d}}{\mathrm{d}X}f_0(W+X)\right]=-\log q\sum_{k=0}^{\infty}\frac{q^{2k}(q^{\theta}-q^{W+X})^2\overline{(1-q^{W+X+k})}}{(1-q^{\theta+k})^2|1-q^{W+X+k}|^2}.$$

Writing $q^{W+X}=q^{\theta}+z'$, the k-th term in the series above has the same sign as

$$\mathfrak{Re}\left[(q^{\theta}-q^{W+X})^2\overline{(1-q^{W+X+k})}\right]=(z'^2\overline{(1-q^k(q^{\theta}+z'))})=z'^2(1-q^{\theta}q^k)-z'|z'|^2q^k. \tag{2.18}$$

If $\mathfrak{Re}\left[q^{W+j}-q^{\theta}\right]\geqslant 0$, then $\arg(z')\leqslant\pi/2$. Moreover, since $W\in\mathcal{C}_{\theta,\pi/4}$, $W+X$ is on the right of $\mathcal{C}_{\theta,\pi/4}$, which exactly means that $\arg(z')\geqslant\pi/4$. Hence both terms in the right-hand-side of (2.18) have negative real part.

Case 2 : $\mathfrak{Re}\left[q^{W+j}-q^{\theta}\right]<0$. This condition geometrically means that $W+j$ lies on the right of $\mathcal{C}_{\theta,\pi/2}$. Now, it may happen that $\mathfrak{Re}[f_0]$ is not decreasing on the horizontal line between W and $W+j$. The idea here, inspired from the proofs of Lemmas 6.10 and 6.12 in [FV13], is to find another path from W to $W+j$ along which $\mathfrak{Re}[f_0]$ is decreasing.

Let $\hat{W}(t)=\log_q(q^{\theta}+e^{i\pi/2\mathrm{sgn}(t)}|t|)$ a parametrization of $\mathcal{C}_{\theta,\pi/2}$. Let t_j be the real number such that $\mathfrak{Re}\left[W+j\right]=\mathfrak{Re}\left[\hat{W}(t_j)\right]$ (see Figure 2.4 and Figure 2.5). Let s_j be the real such that $\mathfrak{Im}\left[W(s_j)\right]=\mathfrak{Im}\left[\hat{W}(t_j)\right]$. From W to $W(s_j)$ along the contour $\mathcal{C}_{\theta,\pi/4}$, $\mathfrak{Re}[f_0]$ is decreasing by steep-descent property of this contour stated in Lemma 2.2.4. From $W(s_j)$ to $\hat{W}(t_j)$ on a horizontal line, $\mathfrak{Re}[f_0]$ is decreasing from the first part of the proof, because for any Z on this line, we have $\mathfrak{Re}\left[q^Z-q^{\theta}\right]\geqslant 0$. It remains to prove that on the vertical line from $\hat{W}(t_j)$ to $W+j$, $\mathfrak{Re}[f_0]$ is decreasing. It is enough to prove that

$$\forall Y\in\left(0,\mathfrak{Im}[W+j-\hat{W}(t_j)]\right),\ \frac{\mathrm{d}}{\mathrm{d}Y}\left(\mathfrak{Re}\left[f_0(\hat{W}(t_j)+iY)\right]\right)<0.$$

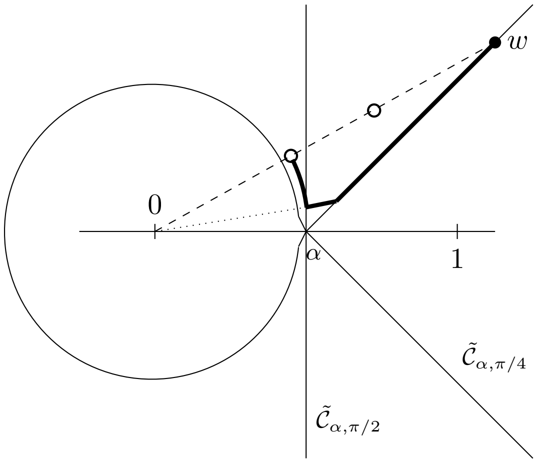

Figure 2.5: Image of Figure 2.4 by the map $x \mapsto q^x$.

Each summand in the series representation for $\frac{\mathrm{d}}{\mathrm{dY}}\left(\mathfrak{Re}\left[f_0(X+iY)\right]\right)$ in Equation (2.16) has the same sign as $|1-q^{\hat{W}(t_j)+iY+k}|^2-(1-q^{\theta+k})^2$. This last quantity is positive when

$$\mathfrak{Re}\left[q^{\hat{W}(t_j)+iY}-q^{\theta}\right]<0.$$

Taking into account the negative prefactor $-\sin(Y\log q)\log q<0$ in Equation (2.16), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}Y}\left(\mathfrak{Re}\left[f_0(\hat{W}(t_j)+iY)\right]\right)<0.$$

It may also happen that $\mathfrak{Re}\left[W+j\right]=\theta+\sigma'$ with $0<\sigma'<2\sigma$, and in this case the path we have just described does not exist. But it suffices to go from W to θ along $\mathcal{C}_{\theta,\pi/4}$, from θ to $\theta+\sigma'$ by a straight horizontal line, and finally to $W+j$ by a vertical line. On the short horizontal line, $\mathfrak{Re}[f_0]$ may increase, but the derivative is bounded, and σ can be chosen as small as we want. Hence the possible increase of $\mathfrak{Re}\left[f_0\right]$ from θ to $\theta+2\sigma$ can be made arbitrarily small, which is enough to prove the Lemma. □

We are now able to prove the following analogue of [FV13, Proposition 6.3].

Proposition 2.2.7. *Asymptotically, the contribution to the Fredholm determinant of the parts of the contours outside any neighbourhood of θ is negligible. More rigorously, for any fixed $\delta>0$ and $\epsilon>0$, there is an N_1 such that for all $N>N_1$*

$$|\det(I+K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\pi/4})}-\det(I+K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_\theta^\delta)}|<\epsilon$$

where $\mathcal{C}_\theta^\delta$ is the truncated contour $\mathcal{C}_{\theta,\pi/4}\cap\{W\ ;\ |W-\theta|\leqslant\delta\}$, and

$$K_{x,\delta}(W,W')=$$
$$\frac{q^W\log q}{2i\pi}\int_{\mathcal{D}_W^\delta}\frac{\mathrm{d}Z}{q^Z-q^{W'}}\frac{\pi}{\sin(\pi(Z-W))}\frac{\exp(Nf_0(Z)+N^{2/3}f_1(Z)+N^{1/3}f_2(Z))}{\exp(Nf_0(W)+N^{2/3}f_1(W)+N^{1/3}f_2(W))}\frac{\phi(Z)}{\phi(W)} \tag{2.19}$$

and analogously $\mathcal{D}_W^\delta=\mathcal{D}_W\cap\{Z\ ;\ |Z-\bar{A}|\leqslant\delta\}$.

Proof. This Proposition is the precise adaptation of Proposition 6.3 in [FV13] and we reproduce the proof done therein. We have the Fredholm determinant expansion

$$\det(I+K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\pi/4})} = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{R}}\mathrm{d}s_1\dots\int_{\mathbb{R}}\mathrm{d}s_k \det\left(K_x\left(W(s_i),W(s_j)\right)_{1\leqslant i,j\leqslant k}\right)\frac{\mathrm{d}W(s_i)}{\mathrm{d}s_i}. \tag{2.20}$$

Let us denote by s_δ the positive real number such that $|W(s_\delta)-\theta| = \delta$. We need to prove that if we replace all the integrations on $\mathbb{R}$ in (2.20) by integrations on $[-s_\delta, s_\delta]$, the error that we make goes to zero when N goes to infinity. We give a dominated convergence argument. Note that the integrable bound in equation (2.17) is not useful here since this bound is valid for a fixed N.

By Lemma 2.2.5 and Lemma 2.2.6 together with the steep-descent properties of the contours, one can find a constant $c_\delta > 0$ such that for any $N > N_0$ and $|s| > s_\delta$,

$$\mathfrak{Re}\left[f_0(Z) - f_0(W(s))\right] < -c_\delta s.$$

Furthermore the integral in (2.19) is absolutely integrable. For the vertical part of the contour $\mathcal{D}_W$, this is due to the exponential decay of the sine in the denominator. Thus, one can find another positive constant C_δ such that for $|s| > s_\delta$ and $N > N_0$, one has

$$|K_x(W(s),W')| < C_\delta \exp\left(-\frac{c_\delta}{2}Ns\right). \tag{2.21}$$

Hence, when $N \geqslant N_0$ the series expansion of the error term, that is the expression in (2.20) with integrations on $\mathbb{R}^k \setminus [-s_\delta, s_\delta]^k$, can be uniformly bounded by a convergent series of absolutely convergent integrals. Thus, by dominated convergence, the error goes to zero.

To conclude the proof of the Proposition, we have to localize the Z integrals on $\mathcal{D}_W^\delta$, and we outline the arguments of [FV13]. The behaviour in the Z variables is $e^{-\pi\mathfrak{Im}[Z]}$ due to the sine in the denominator. Hence by the steep-descent property of the contour for Z on each period, and the same kind of dominated convergence arguments, one can localize the Z integrals in neighbourhoods of size δ around each $\theta + i2k\pi/\log q$ for $k \in \mathbb{Z}$. Moreover one can show that the contribution of the integrals on these δ-neighbourhoods is $\mathcal{O}(N^{-1/3})$ as soon as $k \neq 0$, and summable on k. □

We make the change of variables

$$W = \theta + wN^{-1/3}, W' = \theta + w'N^{-1/3}, Z = \theta + zN^{-1/3}.$$

In order to adapt the rest of the asymptotic analysis made in [FV13], we need the following estimate on the behaviour of our additional factor inside the kernel.

Lemma 2.2.8. *For any z and w, we have*

$$\frac{\phi(\theta + zN^{-1/3})}{\phi(\theta + wN^{-1/3})} \xrightarrow[N\to\infty]{} 1. \tag{2.22}$$

Moreover, there exist constants $c_\phi, C_\phi > 0$ such that for $|Z-\theta| < c_\phi$ and $|W-\theta| < c_\phi$, one has

$$\left|\frac{\phi(Z)}{\phi(W)}\right| \leqslant C_\phi. \tag{2.23}$$

Proof. The infinite product $(z;q)_\infty$ converges uniformly on any disk centred in 0. Here for all $1 \leqslant j \leqslant m$, $q^\theta < \alpha \leqslant a_{i_j}$. Thus, each factor tends to a non null real number, and one can exchange limit and infinite product. The limit does neither depend on z nor on w, and one has

$$\frac{(q^{\theta+zN^{-1/3}}/a_{i_j};q)_\infty}{\left((q^{\theta+zN^{-1/3}};q)_\infty\right)} \underset{N\to\infty}{\longrightarrow} \frac{(q^\theta/a_{i_j};q)_\infty}{((q^\theta;q)_\infty)}. \tag{2.24}$$

The factors in $\phi(\theta+zN^{-1/3})$ and $\phi(\theta+wN^{-1/3})$ compensate in the limit, and

$$\frac{\phi(\theta+zN^{-1/3})}{\phi(\theta+wN^{-1/3})} \underset{N\to\infty}{\longrightarrow} 1.$$

Assuming $|Z-\theta| < c_\phi$ and $|W-\theta| < c_\phi$ where c_ϕ is chosen small enough, $|q^{Z+k}/a_{i_j}|$ is uniformly bounded by a constant smaller than 1. Hence $\phi(Z)$ and $\phi(W)$ are uniformly bounded above and below by positive constants, and one can find a constant C_ϕ so that (2.23) holds. □

Due to the change of variables, we define new integration contours which we choose as straight lines for simplicity. For $L \in \mathbb{R}_+ \cup \{\infty\}$, the contours $\mathcal{C}_{\varphi,L}$ and $\mathcal{D}_{\varphi,L}$ are adapted from [FV13] and defined in the following way: $\mathcal{C}_{\varphi,L} = \{e^{i(\pi-\varphi)\mathrm{sgn}(y)}|y|, |y| \leqslant L\}$ for some angle $\varphi < \pi/4$. Analogously we define $\mathcal{D}_{\varphi,L} = \{e^{i\varphi\mathrm{sgn}(y)}|y|, |y| \leqslant L\}$. This modification of contours can be performed without changing the value of the integral as soon as we keep the same endpoints, and the angle φ and the parameter σ can be chosen so that it is the case.

Proposition 2.2.9. *We have the convergence*

$$\lim_{N\to\infty} \det(I+K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\pi/4})} = \det(I+K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\infty})}$$

where for $L \in \mathbb{R}_+ \cup \{+\infty\}$,

$$K'_{x,L} = \frac{1}{2i\pi}\int_{\mathcal{D}_{\varphi,L}} \frac{\mathrm{d}z}{(z-w')(w-z)} \frac{\exp(\chi z^3/3 + c(\log q)^2 z^2/2 + zc^2(\log q)^4/(4\chi) - zx\chi^{1/3})}{\exp(\chi w^3/3 + c(\log q)^2 w^2/2 + wc^2(\log q)^4/(4\chi) - wx\chi^{1/3})}. \tag{2.25}$$

Proof. For the sake of self-containedness, we reproduce the proofs of Propositions 6.4 to 6.6 of [FV13] which still hold with a slight modification for the pointwise limit.

Consider the rescaled kernel

$$K^N_{x,\delta}(w,w') = N^{-1/3}K_{x,\delta N^{1/3}}(\theta+wN^{-1/3}, \theta+w'N^{-1/3})$$

where we use the contours $\mathcal{C}_{\varphi,\delta N^{1/3}}$ and $\mathcal{D}_{\varphi,\delta N^{1/3}}$. By a simple change of variables,

$$\det(I+K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}^\delta_\theta)} = \det(I+K^N_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\delta N^{1/2}})}.$$

First we estimate the argument in the exponential in (2.19). By Taylor approximation, there exists C_{f_0}, such that for $|Z-\theta| < \theta$,

$$\left| f_0(Z) - f_0(\theta) - \frac{\chi}{3}(Z-\theta)^3 \right| < C_{f_0}|Z-\theta|^4 \tag{2.26}$$

and since $f_1'(\theta)=0$ and $f_1''(\theta)=c(\log q)^2$, there exists C_{f_1}, such that for $|Z-\theta|<\theta$,

$$\left|f_1(Z)-f_1(\theta)-c(\log q)^2(Z-\theta)^2\right|<C_{f_1}|Z-\theta|^3.$$

Let us denote the argument in the exponential in (2.19) as

$$f(Z,W,N):=Nf_0(Z)+N^{2/3}f_1(Z)+N^{1/3}f_2(Z)-Nf_0(W)-N^{2/3}f_1(W)-N^{1/3}f_2(W),$$

and the argument in the exponential in (2.25) as

$$\begin{aligned}f^{\lim}(z,w):=&\left(\chi z^3/3+c(\log q)^2z^2/2+c^2(\log q)^4/(4\chi)z-x\chi^{1/3}z\right)\\&-\left(\chi w^3/3+c(\log q)^2w^2/2+c^2(\log q)^4/(4\chi)w-x\chi^{1/3}w\right).\end{aligned}$$

Using the Taylor approximations above and rescaling the variables, one has that for $w\in\mathcal{C}_{\varphi,\delta N^{1/3}}$, $z\in\mathcal{D}_{\varphi,\delta N^{1/3}}$, and $Z=\theta+zN^{-1/3}, W=\theta+wN^{-1/3}$,

$$\begin{aligned}\left|f(Z,W,N)-f^{\lim}(z,w)\right| &< N^{-1/3}\left(C_{f_0}(|z|^4+|w|^4)+C_{f_1}(|z|^3+|w|^3)\right) &(2.27)\\ &\leqslant \delta\left(C_{f_0}(|z|^3+|w|^3)+C_{f_1}(|z|^2+|w|^2)\right). &(2.28)\end{aligned}$$

Now we estimate the remaining factors in the integrand in (2.19). Let us denote

$$F(Z,W,W'):=\frac{N^{-1/3}}{q^Z-q^{W'}}\frac{\pi}{\sin(\pi(Z-W))}\frac{\phi(Z)}{\phi(W)},$$

and the remaining factors in the integrand in (2.25) as

$$F^{\lim}(z,w,w'):=\frac{1}{z-w'}\frac{1}{z-w}.$$

Let us prove that for any $w,w'\in\mathcal{C}_N$, $K^N_{x,\delta}(w,w')-K'_{x,\delta N^{1/3}}(w,w')$ goes to zero when N goes to infinity. Indeed, the error can be estimated by

$$\begin{aligned}|K^N_{x,\delta}(w,w')-K'_{x,\delta N^{1/3}}(w,w')|<&\int_{\mathcal{D}_N}\mathrm{d}z\exp(f^{\lim})|F(Z,W,W')|\left|\exp(f-f^{\lim})-1\right|\\&+\int_{\mathcal{D}_N}\mathrm{d}z\exp(f^{\lim})\left|F-F^{\lim}\right|, \quad (2.29)\end{aligned}$$

where we have omitted the arguments of the functions $f(Z,W,N)$, $f^{\lim}(z,w)$, $F(Z,W,W')$, $F^{\lim}(z,w,w')$, with $Z=\theta+zN^{-1/3}$ as before, and likewise for W,W'. By estimates (2.27) and (2.28) and the inequality $|\exp(x)-1|\leqslant|x|\exp(|x|)$, we have

$$|\exp(f-f^{\lim})-1|<N^{-1/3}P(|z|,|w|)\exp\left(\delta\left(C_{f_0}(|z|^3+|w|^3)+C_{f_1}(|z|^2+|w|^2)\right)\right),$$

where P is the polynomial $P(X,Y)=C_{f_0}(X^4+Y^4)+C_{f_1}(X^3+Y^3)$. Hence, for δ small enough,

$$\exp(f^{\lim})|\exp(f-f^{\lim})-1|$$

has cubic exponential decay in $|z|$ when z goes to infinity along the contour $\mathcal{D}_\infty$. Hence the first integral in (2.29) goes to zero as N goes to infinity by dominated convergence.

The second integral in (2.29) also goes to zero by dominated convergence since one can bound

$$\left| F(\theta + zN^{-1/3}, \theta + wN^{-1/3}, \theta + w'N^{-1/3}) - F^{\lim}(z, w, w') \right| < \\ N^{-1/3} Q(|z|, |w|, |w'|) F^{\lim}(z, w, w'),$$

for some polynomial Q.

In order to prove that the difference of Fredholm determinants goes to zero as well, one could show that the difference of operators $K^N_{x,\delta}$ and $K'_{x,\delta N^{1/3}}$ acting on $\mathbb{L}^2(\mathcal{C}_\infty)$ goes to zero in trace-class norm, but we give a simpler dominated convergence argument instead. The estimates in right-hand-sides of Equations (2.28) and (2.23) show that $K^N_{x,\delta}$ has cubic exponential decay. More precisely, there exists a constant $C > 0$ independent of N such that for all $w, w' \in \mathcal{C}_{\varphi,\delta N^{1/3}}$,

$$|K^N_{x,\delta}(w, w')| < C \exp\left(f^{\lim}(0, w) + C_{f_0}\delta|w|^3 + C_{f_1}\delta|w|^2 \right).$$

Hence for δ small enough, Hadamard's bound yields

$$\left| \det\left(K^N_{x,\delta}(w_i, w_j)_{1 \leqslant i,j \leqslant n} \right) \right| \leqslant n^{n/2} C^n \prod_{i=1}^{n} e^{\chi/6 \mathfrak{Re}[w_i^3]}.$$

It follows that the Fredholm determinant expansion,

$$\det(I + K^N_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\delta N^{1/3}})} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{C}_{\varphi,\delta N^{1/3}}} \mathrm{d}w_1 \dots \int_{\mathcal{C}_{\varphi,\delta N^{1/3}}} \mathrm{d}w_n \det\left(K^N_{x,\delta}(w_i, w_j)_{1 \leqslant i,j \leqslant n} \right),$$

is absolutely integrable and summable. Thus, by dominated convergence,

$$\lim_{N\to\infty} \det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\pi/4})} = \lim_{N\to\infty} \det(I + K'_{x,\delta N^{1/3}})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\delta N^{1/3}})}.$$

Since the integrand in $K'_{x,\delta N^{1/3}}$ has cubic exponential decay along the contours $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$, dominated convergence, again, yields

$$\det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_{\theta,\pi/4})} \xrightarrow[N\to\infty]{} \det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_\infty)}.$$

□

Now we explain how the limit of the q-Laplace transform characterizes the limit law of the rescaled position of particles. The sequence of functions $E_N(y) := 1/(-q^{-yN^{1/3}}; q)_\infty$ is such that for any $N > 0$, $E_N(y)$ is strictly decreasing with limit 1 when y goes to $-\infty$, and with limit 0 when y goes to $+\infty$. Additionally, for each $\varepsilon > 0$, E_N converges uniformly to $\mathbb{1}_{y \leqslant 0}$ on $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$. Using Lemma 4.39 in [BC14] to replace E_N by its limit and with our choice of ζ,

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(\xi_N < x) &= \lim_{N\to\infty} \mathbb{E}\left[E_N\left(\frac{\chi^{1/3}}{|\log q|}(\xi_N - x) \right) \right] \\
&= \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{(\zeta q^{X_N(t)+N}; q)_\infty} \right] \\
&= \det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\infty})}.
\end{aligned}$$

Finally, using a classical reformulation of the kernel (see [BCF14, Lemma 8.7]) to get the Fredholm determinant of an operator acting on $\mathbb{L}^2(\mathbb{R}_+)$, and after the change of variables $z \leftarrow \chi^{1/3}(z + c(\log q)^2/(2\chi))$ and likewise for w and w',

$$\det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_{\varphi,\infty})} = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)}$$

and we conclude that

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{GUE}}(x).$$

2.2.2 Case $\alpha = q^\theta$, critical value.

The function ϕ introduces a pole of order k in $A = \theta$ in the kernel K_x, for the variable W. The contour of W must enclose this pole, and thus $\mathcal{C}_{\bar{A},\varphi}$ has to pass on the right of θ. The contour can be chosen as in the previous section, except for a modification (e.g. a small circle of radius $(\epsilon/2)N^{-1/3}$ centred at θ) in a $N^{-1/3}$-neighbourhood of θ. In order to stay on the right of $\mathcal{C}_{\bar{A},\varphi}$, the contour $\mathcal{D}_W$ can be simply shifted to the right by $\epsilon N^{-1/3}$. In order to adapt the arguments of the case $\alpha > q^\theta$, we only need the pointwise limit and a uniform bound in a neighbourhood of θ for the factor $\phi(Z)/\phi(W)$ introduced in the kernel.

Lemma 2.2.10. *For any z and w, we have*

$$\frac{\phi(\theta + zN^{-1/3})}{\phi(\theta + wN^{-1/3})} \underset{N\to\infty}{\longrightarrow} \left(\frac{z}{w}\right)^k.$$

Moreover, there exist constants $c'_\phi, C'_\phi > 0$ such that for $|Z - \theta| < c'_\phi$ and $|W - \theta| < c'_\phi$, one has

$$\left|\frac{\phi(Z)}{\phi(W)}\right| < C'_\phi \left|\frac{Z-\theta}{W-\theta}\right|^k. \tag{2.30}$$

Proof. For j such that $a_{i_j} > q^\theta$, the limit in (2.24) still holds. We are left with the k factors for which $a_{i_j} = \alpha$. In this case

$$\frac{(q^{\theta+zN^{-1/3}}/\alpha;q)_\infty}{(q^{\theta+zN^{-1/3}};q)_\infty} = \frac{(q^{zN^{-1/3}};q)_\infty}{(q^{\theta+zN^{-1/3}};q)_\infty} \underset{N\to\infty}{\sim} \frac{(-\log q)zN^{-1/3}(q;q)_\infty}{(q^\theta;q)_\infty}. \tag{2.31}$$

The $N^{-1/3}$ and constant factors in $\phi(Z)$ and $\phi(W)$ compensate in the limit, and we get the result.

Let us prove the bound (2.30). For j such that $a_{i_j} > q^\theta$ the factors $\frac{(q^{\theta+zN^{-1/3}}/a_{i_j};q)_\infty}{(q^{\theta+zN^{-1/3}};q)_\infty}$ are bounded as in Lemma 2.2.8. For the factors for which $a_{i_j} = \alpha$, we use the fact that the function $u \mapsto |(1-q^u)/u|$ is bounded above and below by positive constants on some disc centred in 0 of positive radius r. Choosing $c'_\phi \leqslant r$ and small enough so that Lemma 2.2.8 applies, one gets the result. □

With this Lemma, the local modification of the paths has no influence on any of the bounds given previously for large w and z. Only the pointwise limit of the modified kernel is slightly different and given by the above Lemma. We conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{(\zeta q^{X_N(t)+N};q)_\infty}\right] = \det(I + K'_x)_{\mathbb{L}^2(\mathcal{C}_{\varphi,\infty})}$$

where

$$K'_x = \frac{1}{2i\pi}\int_{\mathcal{D}_{\varphi,\infty}} \frac{\mathrm{d}z}{(z-w')(w-z)} \frac{\exp(\chi z^3/3 + c(\log q)^2 z^2/2 + c^2(\log q)^4/(4\chi) z - x\chi^{1/3} z)}{\exp(\chi w^3/3 + c(\log q)^2 w^2/2 + c^2(\log q)^4/(4\chi) w - x\chi^{1/3} w)}\left(\frac{z}{w}\right)^k. \quad (2.32)$$

The contours $\mathcal{C}_{\varphi,\infty}$ and $\mathcal{D}_{\varphi,\infty}$ are slight modifications of those defined in the previous section. Here, $\mathcal{C}_{\varphi,\infty} = \{\theta + e^{i(\pi-\varphi)\mathrm{sgn}(y)}|y| \;; |y| > N^{-1/3}\epsilon/2\} \cup \{\epsilon/2N^{-1/3}e^{i\gamma} \;;\; \gamma \in [\varphi - \pi; \pi - \varphi]\}$. The contour $\mathcal{D}_{\phi,\infty}$ can be chosen as $\{\epsilon N^{-1/3} + e^{i\varphi \mathrm{sgn}(y)}|y|,\ y \in \mathbb{R}\}$.

We reformulate the kernel as a Fredholm determinant acting on $\mathbb{L}^2(\mathbb{R}_+)$ (see [BCF14, Lemma 8.7]), and after the change of variables $z \leftarrow \chi^{1/3}(z + c(\log q)^2/(2\chi))$ and likewise for w and w', we conclude that

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = \det(I - K''_x(w,w'))_{\mathbb{L}^2(x,+\infty)}$$

where

$$K''_x(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{-2i\pi/3}\infty}^{e^{2i\pi/3}\infty} \mathrm{d}w \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \mathrm{d}z \frac{e^{z^3/3 - zu}}{e^{w^3/3 - wv}} \frac{1}{z-w}\left(\frac{z - \frac{c(\log q)^2}{2\chi^{2/3}}}{w - \frac{c(\log q)^2}{2\chi^{2/3}}}\right)^k,$$

where the contour for w passes to the right of $b := \frac{c(\log q)^2}{2\chi^{2/3}}$, and the contours for z and w do not intersect. Finally, by Definition 2.1.5,

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{BBP,k,}\mathbf{b}}(x),$$

with $\mathbf{b} = (b, \dots, b)$.

Remark 2.2.11. In the case where for $1 \leqslant i \leqslant k$, $a_i = q^{\theta + \tilde{b}_i N^{-1/3}}$ and the rates of all other particles are higher than q^θ, Lemma 2.2.10 still applies and the factor $(z/w)^k$ in Equation (2.32) has to be replaced by $\prod_{i=1}^k (z-b_i)/(w-b_i)$. Then $((z-b)/(w-b))^k$ with $b = c(\log q)^2\chi^{-2/3}/2$ gets replaced by $\prod_{i=1}^k (z-b_i)/(w-b_i)$ with $b_i = b + \tilde{b}_i$, and finally

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{\mathrm{BBP,k,}\mathbf{b}}(x),$$

with $\mathbf{b} = (b_1, \dots, b_k)$.

2.2.3 Case $\alpha < q^\theta$, Gaussian fluctuations

We start again from the result of Theorem 2.2.2. One cannot use the same contour for the Fredholm determinant, because the pole for $W = A$ in $K_x(W, W')$ has to be inside the contour $\mathcal{C}_{\bar{A},\varphi}$, which means $\bar{A} \geqslant A > \theta$. Let us choose

$$\zeta = -q^{-gN - cN^{1/2} - \sigma^{1/2}\frac{N^{1/2}}{\log q}}$$

so that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{X_N(\tau^*(N,c)) - p^*(N,c)}{N^{1/2}\sigma^{1/2}/(\log q)} < x\right) = \lim_{N\to\infty} \mathbb{E}\left[1/(\zeta q^{X_N(\tau)}; q)_\infty\right]$$

with the new macroscopic position $p^*(N,c) = (g-1)N + cN^{1/2}$.

Again, $\det(I+\tilde{K}_\zeta)_{\mathbb{L}^2(\tilde{\mathcal{C}}_{\tilde{\alpha},\varphi})} = \det(I+K_x)_{\mathbb{L}^2(\mathcal{C}_{\tilde{A},\varphi})}$ where

$$K_x(w,w') = \frac{q^W \log q}{2i\pi} \int_{\mathcal{D}_W} \frac{\mathrm{d}Z}{q^Z - q^{W'}} \frac{\pi}{\sin(\pi(Z-W))} \frac{\exp(Ng_0(Z) + N^{1/2}g_1(Z))}{\exp(Ng_0(W) + N^{1/2}g_1(W))} \frac{\phi(Z)}{\phi(W)} \tag{2.33}$$

with

$$\begin{aligned} g_0(Z) &= -g\log(q)Z + \kappa q^Z + \log(q^Z;q)_\infty, \\ g_1(Z) &= -Z\log(q)c - \sigma^{1/2}xZ + \frac{c}{\alpha}q^Z. \end{aligned}$$

The asymptotic behaviour is governed by the real part of the function g_0. By a direct calculation and Equations (2.7) and (2.3), one has that

$$\begin{aligned} g_0'(Z) &= -g\log(q) + \log(q)\kappa q^Z + \sum_{k=0}^{\infty} \frac{-\log(q)q^{Z+k}}{1-q^{Z+k}} \\ &= \frac{\Psi_q'(\theta)}{q^\theta \log(q)}(q^Z - \alpha) + \Psi_q(A) - \Psi_q(Z), \\ g_0''(Z) &= \Psi_q'(\theta)q^{Z-\theta} - \Psi_q'(Z). \end{aligned}$$

We see immediately that $g_0'(A) = 0$, and using the series representation (2.12) and (2.13), for $A > \theta$,

$$g_0''(A) = \sigma = (\log q)^2 \sum_{k=0}^{\infty} q^{A+k} \left(\frac{1}{(1-q^{\theta+k})^2} - \frac{1}{(1-q^{A+k})^2} \right) > 0. \tag{2.34}$$

Lemma 2.2.12. *1. The contour $\mathcal{C}_{A,\pi/4}$ is steep-descent for $-\mathfrak{Re}[g_0]$ in the sense that the function $s \mapsto \mathfrak{Re}[g_0(W(s))]$ is increasing for $s \geqslant 0$, where $W(s)$ is a parametrization of $\mathcal{C}_{A,\pi/4}$.*

2. The function $\mathfrak{Re}[g_0]$ is periodic on $\{A+\sigma+i\mathbb{R}\}$ with period $2\pi/|\log q|$. Moreover, $t \mapsto \mathfrak{Re}[g_0(A+\sigma+it)]$ is decreasing on $[0, -\pi/\log q]$ and increasing on $[\pi/\log q, 0]$, for any $\sigma > 0$.

Proof. 1. We assume that $\alpha < q^\theta$. Using the parametrization of the contour $\mathcal{C}_{A,\varphi}$ $W(s) = \log_q(\alpha + se^{i\varphi})$ as before , we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{Re}[g_0(W(s))]\right) = \sum_{k=0}^{\infty} \Bigg(\frac{sq^k}{(1-q^{\theta+k})^2} \frac{\alpha\cos(2\varphi) + s\cos(\varphi)}{|\alpha + se^{i\varphi}|^2} + \frac{\alpha q^k}{(1-\alpha q^k)} \frac{\alpha\cos(\varphi) + s}{|\alpha + se^{i\varphi}|^2} - \frac{(\cos(\varphi) - (\alpha\cos(\varphi)+s)q^k)q^k}{|1-(\alpha+se^{i\varphi})q^k|^2} \Bigg).$$

For $\varphi \leqslant \pi/4$, using the fact that $q^\theta > \alpha$, and factoring the summand, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{Re}[g_0(W(s))]\right) > \sum_{k=0}^{\infty} \frac{q^{2k}s^2\left(q^k s^2\cos(\varphi) - (1-2\alpha q^k)s\cos(2\varphi) - \alpha(1-\alpha q^k)\cos(3\varphi)\right)}{(1-\alpha q^k)^2|\alpha+se^{i\varphi}|^2|1-(\alpha+se^{i\varphi})q^k|^2},$$

which is positive for $s > 0$ and $\varphi = \pi/4$.

2. Let $Z(t) = A + \sigma + it$. Notice that $g_0'(Z) = -\log(q)(g-f) + f_0'(Z)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{Re}\left[g_0(Z(t))\right]\right) = -\mathfrak{Im}\left[g_0'(Z(t))\right].$$

By Lemma 2.2.4,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{Re}\left[g_0(Z(t))\right]\right) = -\sin(t\log q)\log q\sum_{k=0}^{\infty} q^{A+\sigma+k}\left(\frac{1}{(1-q^{\theta+k})^2} - \frac{1}{|1-q^{A+\sigma+it+k}|^2}\right)$$

has the same sign as $\sin(t\log q)$, proving the steep-descent property.

∎

Lemma 2.2.13. *The kernel $K_x(W(s), W')$ has exponential decay, in the sense that there exist N_0, $s_0 \geqslant 0$ and $c > 0$ such that for all $s > s_0$ and $N > N_0$,*

$$|K_x(W(s), W')| \leqslant \exp(-cNs).$$

Proof. This Lemma is very similar with [FV13, Lemma 6.10] and we adapt the proof.

We first estimate the contribution of the integration along the vertical line $A+\sigma+i\mathbb{R}$. For any $\varphi \in (0, \pi/4]$,

$$\lim_{s\to+\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{Re}\left[g_0(W(s))\right]\right) = \kappa > 0.$$

Therefore, for s large enough

$$\mathfrak{Re}\left[Ng_0(W(s)) + N^{1/2}g_1(W(s))\right] > \kappa sN/2 - N^{1/2}\sigma^{1/2}x|\log_q(|\alpha+s+is|)|/2.$$

Thus, one can find N_0 and $s_1 \geqslant 0$ such that for all $s > s_1$ and $N > N_0$, $\exp(-Ng_0(W(s)) - N^{1/2}g_1(W(s))) < \exp(-\kappa Ns/4)$. As the vertical line is at a distance at least $\sigma/2$ from the poles coming from the sine, the factor $\frac{\pi}{\sin(\pi(Z-W))}$ is bounded by $Ce^{-\pi\mathfrak{Im}[Z]}$ for some constant $C > 0$. The remaining factors in the integrand are bounded for $W \in \mathcal{C}_{A,\pi/4}$ and $Z \in \mathcal{D}_W$.

Now we estimate the contribution of the integration along the small circles $\mathcal{E}_1, \dots, \mathcal{E}_{k_W}$. It is enough to prove that each residue at the poles in $W(s)+1, \dots, W(s)+k_{W(s)}$ is at most $\exp(-cNs)$, as the number of poles is only logarithmic in s. Instead of reproducing word-for-word the proof of Lemma 6.10 in [FV13], observe that

$$\begin{aligned}\mathfrak{Re}\left[g_0(W)\right] - \mathfrak{Re}\left[g_0(W+j)\right] &= \left(\mathfrak{Re}\left[g_0(W)\right] - \mathfrak{Re}\left[f_0(W)\right]\right) \\ &+ \left(\mathfrak{Re}\left[f_0(W)\right] - \mathfrak{Re}\left[f_0(W+j)\right]\right) + \left(\mathfrak{Re}\left[f_0(W+j)\right] - \mathfrak{Re}\left[g_0(W+j)\right]\right). \quad (2.35)\end{aligned}$$

The sum of the first and third terms is just $-\log(q)(f-g)j$ which is positive. And by the arguments of [FV13, Lemma 6.10], for $W = W(s)$ with large s, there exists a constant $c' > 0$ such that for $s \geqslant s_2$

$$\mathfrak{Re}\left[f_0(W)\right] - \mathfrak{Re}\left[f_0(W+j)\right] > c's.$$

One concludes that the integrand in 2.33 behaves like $\exp(-cNs)$ which concludes the proof. ∎

As in the case $\alpha > q^\theta$, we can now formulate the Fredholm determinant on the contour $\mathcal{C}_{A,\pi/4}$ (instead of $\mathcal{C}_{A,\varphi}$ for $\varphi \in (0, \pi/4)$). As in Section 2.2.2, a small modification of the contours on a $N^{-1/2}$-neighbourhood of A is needed so that the pole for W in A is inside the contour, and the contour $\mathcal{C}_{A,\pi/4}$ stays to the left of $\mathcal{D}_{\mathcal{W}}$.

Proposition 2.2.14. *For any fixed $\delta > 0$ and $\epsilon > 0$, there is an N_1 such that*

$$|\det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_{A,\pi/4})} - \det(I + K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_A^\delta)}| < \epsilon$$

for all $N > N_1$ where $\mathcal{C}_A^\delta = \mathcal{C}_{A,\pi/4} \cap \{W \ \ |W - A| \leqslant \delta\}$, and

$$K_{x,\delta}(W, W') = \frac{q^W \log q}{2i\pi} \int_{\mathcal{D}_W^\delta} \frac{\mathrm{d}Z}{q^Z - q^{W'}} \frac{\pi}{\sin(\pi(Z - W))} \frac{\exp(Ng_0(Z) + N^{1/2}g_1(Z))}{\exp(Ng_0(W) + N^{1/2}g_1(W))} \frac{\phi(Z)}{\phi(W)} \tag{2.36}$$

and $\mathcal{D}_W^\delta = \mathcal{D}_W \cap \{Z \ \ |Z - A| \leqslant \delta\}$

Proof. Using Lemma 2.2.12 and Lemma 2.2.13, one can apply exactly the same proof as in Proposition 2.2.7. We are left with proving that the contribution of all small circles in the contour $\mathcal{D}_W$ goes to zero when N tends to infinity, which results from the following Lemma as in Proposition 2.2.7.

Lemma 2.2.15. *There exists $\eta > 0$ such that for any $W \in \mathcal{C}_{A,\pi/4}$,*

$$\mathfrak{Re}\left[g_0(W) - g_0(W + j)\right] > \eta,$$

for all $j = 1, \dots, k_W$.

Proof. First notice that

$$\mathfrak{Re}\left[g_0(W) - g_0(W + j)\right] = (-\log q)\left(f(q, \theta) - g(q, \theta)\right) j + \mathfrak{Re}\left[f_0(W) - f_0(W + j)\right].$$

As $f(q, \theta) - g(q, \theta) > 0$, it is enough to prove that for any $W \in \mathcal{C}_{A,\pi/4}$,

$$\mathfrak{Re}\left[f_0(W) - f_0(W + j)\right] \geqslant 0,$$

for all $j = 1, \dots, k_W$. The proof is adapted from Lemma 2.2.6 and splits here into three parts. As before, we prove the result for the residues lying above the real axis. Let W_A be the point where $\mathcal{C}_{A,\pi/4}$ and $\mathcal{C}_{\theta,\pi/2}$ intersect (above the real axis). In other words, $W_A = \log_q(q^\theta + i(q^\theta - \alpha)) = \log_q(\alpha + (1 + i)(q^\theta - \alpha))$.

Case 1 : $\mathfrak{Re}\left[q^{W+j}\right] > q^\theta$. This is the case when $W + j$ lies on the left of $\mathcal{C}_{\theta,\pi/2}$. The fact that $\mathfrak{Re}\left[f_0(W) - f_0(W + j)\right] \geqslant 0$ was proved in Lemma 2.2.6.

Case 2 : $\mathfrak{Re}\left[q^{W+j}\right] < q^\theta$ and $\mathfrak{Re}[W + j] \leqslant \mathfrak{Re}\left[W_A\right]$. Let $W(s)$ be a parametrization of $\mathcal{C}_{A,\pi/4}$ and $\hat{W}(t)$ a parametrization of $\mathcal{C}_{\theta,\pi/2}$. Let t_j be such that $\mathfrak{Re}\left[\hat{W}(t_j)\right] = \mathfrak{Re}\left[W + j\right]$, and u_j be such that $\mathfrak{Im}\left[W(u_j)\right] = \mathfrak{Im}\left[\hat{W}(t_j)\right]$ (see Figure 2.6). If $\mathfrak{Re}\left[W + j\right] < \mathfrak{Re}\left[W_A\right]$, we consider the path from W to $W(u_j)$ along $\mathcal{C}_{A,\pi/4}$, from $W(u_j)$ to $\hat{W}(t_j)$ along a horizontal line, and from $\hat{W}(t_j)$ to $W + j$ along a vertical line. The fact that the horizontal segment is on the left of $\mathcal{C}_{\theta,\pi/2}$ and the vertical segment is on the right of $\mathcal{C}_{\theta,\pi/2}$ ensures, by the same arguments as in Lemma 2.2.6, that $\mathfrak{Re}\left[f_0\right]$ decays along this path.

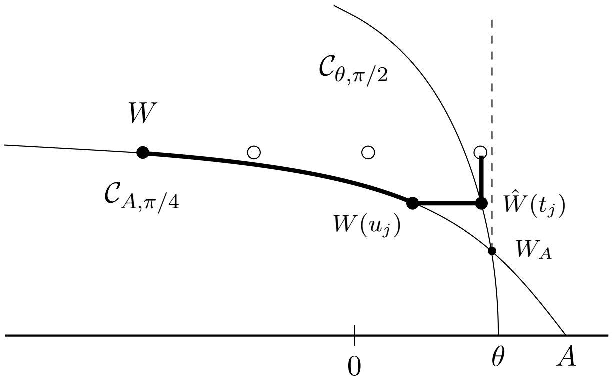

Figure 2.6: The thick line is the path from W to $W+j$ with $j=3$ in the case 2 : $\mathfrak{Re}\left[q^{W+j}\right] < q^{\theta}$ and $\mathfrak{Re}[W+j] \leqslant \mathfrak{Re}\left[W_A\right]$.

Case 3 : $\mathfrak{Re}\left[W_A\right] < \mathfrak{Re}\left[W+j\right] \leqslant A$. Let s_j be such that $\mathfrak{Re}\left[W(s_j)\right] = \mathfrak{Re}\left[W+j\right]$. We consider the path from W to $W(s_j)$ along $\mathcal{C}_{A,\pi/4}$, and from $W(s_j)$ to $W+j$ along a vertical line. The fact that the vertical segment from $W(s_j)$ to $W+j$ is on the right of $\mathcal{C}_{\theta,\pi/2}$ ensures again that $\mathfrak{Re}\left[f_0\right]$ decays along this path.

It may also happen that $\mathfrak{Re}\left[W+j\right] = A+\sigma'$ for some $0<\sigma'\leqslant 2\sigma$, but we treat this case exactly as in the end of the proof of Lemma 2.2.6. □

□

For simplicity, we modify again the contours, the curves becoming straight lines. By the Cauchy theorem, this modification is authorized as soon as the endpoints of the contours coincide. The contour $\mathcal{C}_A^{\delta}$ becomes

$$\left\{A + e^{i(\pi-\gamma)\mathrm{sgn}(y)}|y|, y \in \pm[N^{-1/2},\delta]\right\} \cup \left\{N^{-1/2}e^{it}, t\in[\gamma-\pi,\pi-\gamma]\right\}$$

where the angle $\gamma < \pi/4$ is chosen so that the endpoints coincide. We also consider the corresponding contour $\mathcal{C}_N$ for the rescaled variables $w = N^{1/2}(W-A)$.

Similarly, the contour for the variable Z becomes

$$\left\{A + e^{i(\pi/2-\gamma)\mathrm{sgn}(y)}|y|, y \in \pm[N^{-1/2},\delta]\right\} \cup \left\{N^{-1/2}e^{it}, t\in[-\pi/2+\gamma,\pi/2-\gamma]\right\},$$

and the constant σ used in the definition of $\mathcal{D}_W$ is chosen so that the endpoints coincide. We also consider the corresponding contour $\mathcal{D}_N$ for the rescaled variable $z = N^{1/2}(Z-A)$.

Proposition 2.2.16. *There exist $\delta'>0$ such that for $\delta<\delta'$,*

$$\lim_{N\to\infty} |\det(I+K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_A^{\delta})} - \det(I+K'_{x,N})_{\mathbb{L}^2(\mathcal{C}_N)}| = 0$$

where

$$K'_{x,N}(w,w') = \frac{1}{2i\pi}\int_{\mathcal{D}_N} \frac{\mathrm{d}z}{(w-z)(z-w')} \frac{\exp(\sigma z^2/2 - \sigma^{1/2}zx)}{\exp(\sigma w^2/2 - \sigma^{1/2}wx)} \left(\frac{w}{z}\right)^k. \tag{2.37}$$

Proof. Consider the rescaled kernel

$$K^N_{x,\delta}(w,w') = N^{-1/2} K_{x,\delta N^{1/2}}(A + wN^{-1/2}, A + w'N^{-1/2})$$

where we use the new contour for Z in the definition of $K_{x,\delta N^{1/2}}$, i.e. $A + N^{-1/2}\mathcal{D}_N$. By the previous discussion on the contours,

$$\det(I + K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}^\delta_A)} = \det(I + K^N_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_N)}.$$

We first give an estimate for the exponential factor in the kernel $K^N_{x\delta}$ in Equation (2.36). By Taylor approximation, there exists C_{g_0} such that for $|Z - A| < A$,

$$|g_0(Z) - g_0(A) - \frac{\sigma}{2}(Z - A)^2| < C_{g_0}|Z - A|^3$$

and since $g_1'(A) = -\sigma^{1/2}x$, there exists C_{g_1} such that

$$|g_1(Z) - g_1(A) + \sigma^{1/2}x(Z - A)| < C_{g_1}|Z - A|^2.$$

The argument in the exponential in (2.36) is

$$g(Z, W, N) := N(g_0(Z) - g_0(W)) + N^{1/2}(g_1(Z) - g_1(W)).$$

Let us denote $g^{\lim}(z, w) = \frac{\sigma}{2}(z^2 - w^2) - \sigma^{1/2}x(z - w)$. Using the Taylor expansions above and using the change of variables $Z = A + zN^{-1/2}$, and likewise for W and W', one has

$$\left|g(Z, W, N) - g^{\lim}(z, w)\right| < N^{-1/2}\left(C_{g_0}(|z|^3 + |w|^3) + C_{g_1}(|z|^2 + |w|^2)\right). \tag{2.38}$$

For $z \in \mathcal{D}_N$ and $w \in \mathcal{C}_N$, the last inequality rewrites

$$\left|g(Z, W, N) - g^{\lim}(z, w)\right| < \delta\left(C_{g_0}(|z|^2 + |w|^2) + C_{g_1}(|z| + |w|)\right). \tag{2.39}$$

Now we estimate the remaining factors in the integrand in (2.36). Let us denote

$$G(Z, W, W') := \frac{N^{-1/2}}{q^Z - q^{W'}} \frac{\pi}{\sin(\pi(Z - W))} \frac{\phi(Z)}{\phi(W)},$$

and the remaining factors in the integrand in (2.37) as

$$G^{\lim}(z, w, w') := \frac{1}{z - w'} \frac{1}{z - w} \left(\frac{w}{z}\right)^k.$$

We prove first that for any $w, w' \in \mathcal{C}_N$, $K^N_{x,\delta}(w, w') - K'_{x,N}(w, w')$ goes to zero when N goes to infinity. The error can be estimated by

$$\begin{aligned}|K^N_{x,\delta}(w, w') - K'_{x,N}(w, w')| \leqslant &\int_{\mathcal{D}_N} \mathrm{d}z \exp(g^{\lim})|G(Z, W, W')| \left|\exp(g - g^{\lim}) - 1\right| \\ &+ \int_{\mathcal{D}_N} \mathrm{d}z \exp(g^{\lim}) \left|G - G^{\lim}\right|,\end{aligned} \tag{2.40}$$

where we have omitted the arguments of the functions $g(Z, W, N)$, $g^{\lim}(z, w)$, $G(Z, W, W')$, $G^{\lim}(z, w, w')$, with $Z = A + zN^{-1/2}$ as before, and likewise for W, W'. By (2.38) and (2.39) and the inequality $|\exp(x) - 1| \leqslant |x| \exp(|x|)$, we have

$$|\exp(g - g^{\lim}) - 1| < N^{-1/2} P(|z|, |w|) \exp\left(\delta\left(C_{g_0}(|z|^2 + |w|^2) + C_{g_1}(|z| + |w|)\right)\right),$$

where P is the polynomial $P(X, Y) = C_{g_0}(X^3 + Y^3) + C_{g_1}(X^2 + Y^2)$. Hence for δ small enough, the first integral in (2.40) has quadratic exponential decay (due to the decay of $\exp(g^{\lim})$). Thus, by dominated convergence, the first integral in (2.40) goes to zero as N goes to infinity by dominated convergence. Using estimate (2.30) in Lemma 2.2.10, the second integral in (2.40) also goes to zero since

$$\left|G(A + zN^{-1/2}, A + wN^{-1/2}, A + w'N^{-1/2}) - G^{\lim}(z, w, w')\right| < \\ N^{-1/3} Q(|z|, |w|, |w'|) G^{\lim}(z, w, w'),$$

for some polynomial Q.

Moreover, the estimates in right-hand-sides of Equations (2.39) and (2.30) show that there exists a constant $C > 0$ independent of N such that for all $w, w' \in \mathcal{C}_N$,

$$|K^N_{x,\delta}(w, w')| < C \exp\left(-\sigma/2w^2 + C_{g_0}\delta|w|^2 + \sigma^{1/2} x w + C_{g_1}\delta|w|\right).$$

Hence for δ small enough, Hadamard's bound yields

$$\left|\det\left(K^N_{x,\delta}(w_i, w_j)_{1\leqslant i,j\leqslant n}\right)\right| \leqslant n^{n/2} C^n \prod_{i=1}^{n} e^{-\sigma/4\Re\mathfrak{e}[w_i^2]}.$$

It follows that the Fredholm determinant expansion,

$$\det(I + K^N_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_N)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{C}_N} \mathrm{d}w_1 \dots \int_{\mathcal{C}_N} \mathrm{d}w_n \det\left(K^N_{x,\delta}(w_i, w_j)_{1\leqslant i,j\leqslant n}\right),$$

is absolutely integrable and summable. The conclusion of the Proposition follows by dominated convergence. □

Finally, since the integrand has quadratic exponential decay along the contours $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$, dominated convergence, again, yields

$$\det(I + K'_{x,N})_{\mathbb{L}^2(\mathcal{C}_N)} \underset{N\to\infty}{\longrightarrow} \det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_\infty)}.$$

The third part of Theorem 2.1.6 now follows from a reformulation of the Fredholm determinant achieved in the following Proposition.

Proposition 2.2.17.

$$\det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_\infty)} = G_k(x)$$

where G_k is defined in definition 2.1.5.

Proof. Using the identity,

$$\frac{1}{z-w} = \int_{\mathbb{R}_+} \mathrm{d}\lambda e^{-\lambda(z-w)},$$

valid when $\mathfrak{Re}[z-w] > 0$, the operator $K'_{x,\infty}$ can be factorized. $K'_{x,\infty}(w,w') = -(AB)(w,w')$ where $A : \mathbb{L}^2(\mathbb{R}_+) \to \mathbb{L}^2(\mathcal{C}_\infty)$ and $B : \mathbb{L}^2(\mathcal{C}_\infty) \to \mathbb{L}^2(\mathbb{R}_+)$ are Hilbert–Schmidt operators having kernels

$$A(w,\lambda) = e^{-w^2/2+w(x+\lambda)} w^k \qquad \text{and} \qquad B(\lambda, w') = \frac{1}{2i\pi} \int_{\mathcal{D}_\infty} \frac{\mathrm{d}z}{z^k} \frac{e^{z^2/2-z(x+\lambda)}}{z-w'}.$$

We also have

$$BA(\lambda,\lambda') = \frac{1}{2i\pi} \int_{\mathcal{C}_\infty} \mathrm{d}w B(\lambda,w) A(w,\lambda') = H_k(\lambda+x, \lambda'+x).$$

Since $\det(I-AB)_{\mathbb{L}^2(\mathcal{C}_\infty)} = \det(I-BA)_{\mathbb{L}^2(\mathbb{R}_+)} = G_k(x)$, we get the result. □

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Chapter 3

The q-Hahn asymmetric exclusion process

This chapter is mostly based on the preprint [BC15a], written in collaboration with Ivan Corwin. The section 3.2.2 is different from the submitted version of [BC15a], and contains material from [Bar14].

[BC15a] G. Barraquand and I. Corwin, *The q-Hahn asymmetric exclusion process*, arXiv preprint arXiv:1501.03445 (2015).

[Bar14] G. Barraquand, *A short proof of a symmetry identity for the q-Hahn distribution*, Electron. Commun. in Probab. **19** (2014), no. 50, 1–3.

Contents

3.1 Introduction

The purpose of this chapter is to introduce new families of Bethe ansatz integrable exclusion and zero-range processes on the one-dimensional lattice $\mathbb{Z}$. Our construction generalizes the q-Hahn Boson (zero-range) process introduced in [Pov13] and the q-Hahn TASEP further studied in [Cor14], by allowing jumps in both directions. Under mild assumptions on the microscopic dynamics, such random particle systems are expected to lie in the KPZ universality class. In particular, when started from step initial data, the positions of particles in the bulk of the rarefaction fan are expected to fluctuate according to Tracy-Widom type statistics, up to scaling constants depending on microscopic dynamics. Presently, universality predictions can be confirmed only for a small number of exactly solvable models. A greater variety of well-understood integrable models, with more and more degrees of freedom, is certainly useful towards the study of interacting particle systems under general assumptions. Another application of the study of integrable models is to better understand the cases which are not covered by the KPZ scaling theory. For instance, for an exclusion process starting from step initial data, the statistics of the location of the first particle does not yet fit into a universal framework.

The q-Hahn TASEP is a discrete-time exclusion process on $\mathbb{Z}$, depending on three parameters $q \in (0, 1)$ and $0 \leqslant \nu < \mu < 1$. Each particle jumps independently, and chooses randomly its next location on the right, according to a discrete probability distribution with parameters (q, μ, ν) (see (3.11) for the expression of the weights $\varphi_{q,\mu,\nu}(j|m)$). This distribution is very similar to the weight function for the q-Hahn orthogonal polynomials, hence the name. The main reason why the solvability of the q-Hahn TASEP extends to the partially asymmetric case we consider is that many properties of the transition matrix are preserved by inversion of the parameters q, μ, ν. By taking a limit when μ goes to ν and rescaling the time, the resulting partially asymmetric process is solvable via Bethe ansatz. One obtains closed formulas for the expectation of observables such as $q^{x_n(t)}$, where $x_n(t)$ is the position of the n^{th} particle at time t, using techniques developed by Borodin and Corwin in the context of Macdonald processes [BC14]. Further following those techniques, one arrives at Fredholm determinant formulas for the distribution of $x_n(t)$, which can be analysed asymptotically.

Main results

In order to give an overview of our results, let us introduce our main model in an informal setting. A more precise definition of the process, that we call the (continuous time) *q-Hahn asymmetric exclusion process* (q-Hahn AEP) as well as a discussion about its existence is provided in Section 3.3.1.

For any $q \in (0, 1)$ and $0 \leqslant \nu < 1$ and asymmetry parameters $R, L \geqslant 0$ with $R + L = 1$, the q-Hahn AEP is a continuous-time Markov process on configurations of particles

$$+\infty = x_0(t) > x_1(t) > x_2(t) > \cdots > x_n(t) > \dots \ ; \ x_i \in \mathbb{Z}.$$

The n^{th} particle, locationed at $x_n(t)$ jumps on the right to the location $x_n(t) + j$ at rate (i.e. according to independent exponentially distributed waiting times with rate) $\phi^R_{q,\nu}(j|x_{n-1}(t)-x_n(t)-1)$ for all $j \in \{1, \dots, x_{n-1}(t)-x_n(t)-1\}$, and jumps on the left to the location $x_n(t)-j'$ at rate $\phi^L_{q,\nu}(j'|x_n(t)-x_{n+1}(t)-1)$ for all $j' \in \{1, \dots, x_n(t)-x_{n+1}(t)-1\}$. Figure 3.1 shows two possible jumps for $x_n(t)$. The rates $\phi^R_{q,\nu}(j|m)$ and $\phi^L_{q,\nu}(j|m)$, defined

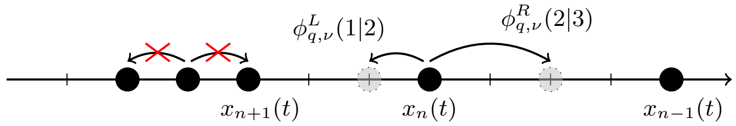

Figure 3.1: Two admissible jumps for the nth particle in the q-Hahn asymmetric exclusion process.

for all integers $1 \leqslant j \leqslant m$, are not arbitrary. To ensure the exact solvability of the process, they are constructed as limits of the q-Hahn distribution:

$$\phi^R_{q,\nu}(j|m) := R\ \frac{\nu^{j-1}}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}},$$
$$\phi^L_{q,\nu}(j|m) := L\ \frac{1}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}}.$$

The q-Pochhammer symbol $(a;q)_n$ is defined in Section 3.2.1. Note that the superscript R (resp. L) on $\phi^R_{q,\nu}$ (resp. $\phi^L_{q,\nu}$) is not an exponent. It only highlights the dependency on the asymmetry parameters R, L. The reader is referred to Section 3.3.1 for a further discussion on the definition of the q-Hahn AEP and the expression for the rates above. Before stating our main formulas regarding this model, we briefly introduce two degenerations.

Setting $\nu = 0$, if $L = 0$, the rates of jumps to the right have the simple form

$$\phi^R_{q,\nu}\big(j|x_{n-1}(t) - x_n(t) - 1\big) = (1 - q^{x_{n-1}(t)-x_n(t)-1})\mathbb{1}_{\{j=1\}},$$

matching those of q-TASEP [BC14]. A further limit when the parameter q goes to zero leads to the well-studied totally asymmetric simple exclusion process (TASEP). However, when $L > 0$, jumps on the left are long-range. Hence our two-sided dynamics are different from those of the classical asymmetric simple exclusion process, but rather generalize the PushASEP [BF08].

Setting $\nu = q$, the rates of jumps no longer depend on the distance to the neighbouring particles. The nth particle jumps on the right to the location $x_n(t) + j$ at rate $R/[j]_{q^{-1}}$ and jumps on the left to the location $x_n(t) - j'$ at rate $L/[j']_q$, where $[j]_{q^{-1}}$ and $[j']_q$ are q-deformed integers (see Section 3.2.1). An example of some possible jumps is shown in Figure 3.2. One of our motivations for studying this model is that it has been known to be exactly solvable for a long time. Indeed, Sasamoto and Wadati [SW98b] introduced a one-parametric family of zero-range processes diagonalizable via Bethe ansatz, called the multi-particle asymmetric diffusion model (MADM). Using a classical coupling between zero-range and exclusion processes that maps the gaps between consecutive particles $x_i - x_{i+1} - 1$ in the exclusion process with the population of the i^{th} site in the zero-range process, the MADM corresponds to the q-Hahn AEP with $R = q/(1+q)$ and $L = 1/(1+q)$ (and $\nu = q$). It was later extended to arbitrary asymmetry parameters $R, L > 0$ [AKK99], and further studied in [Lee12]. We call this model the *MADM exclusion process*. Until now, no formulas amenable to asymptotic analysis have been written down for these systems.

Referring to the general q, ν setting, we also introduce in Section 3.3.1 a *q-Hahn asymmetric zero-range process* (q-Hahn AZRP) on $\mathbb{Z}$ with a finite number of particles. The dynamics are defined in order to correspond to the q-Hahn AEP via exclusion/zero-range

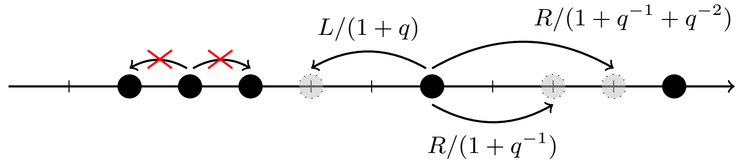

Figure 3.2: Rates of a few admissible jumps in the exclusion process corresponding to the multi-particle asymmetric diffusion model (MADM exclusion process).

transformation. Owing to a Markov duality beetween the q-Hahn AEP and the q-Hahn AZRP, and the Bethe ansatz solvability of the q-Hahn AZRP, we are able to prove the following moment formula for the locations of particles in the exclusion process.

Proposition 3.1.1. *Fix $q \in (0,1)$, $0 \leqslant \nu < 1$, and an integer k. Consider the continuous time q-Hahn AEP started from step initial data (i.e. $x_n(0) = -n$ for $n \geq 1$). Then for any $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$,*

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - q z_B}$$
$$\prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \exp\left((q-1)t\left(\frac{R z_j}{1-\nu z_j} - \frac{L z_j}{1-z_j}\right)\right) \frac{dz_j}{z_j(1-\nu z_j)}. \quad (3.1)$$

where the integration contours $\gamma_1, \dots, \gamma_k$ are chosen so that they all contain 1, γ_A contains $q\gamma_B$ for $B > A$ and all contours exclude 0 and $1/\nu$.

Following the techniques of [BC14] we deduce from Proposition 3.1.1 the following theorem, which provides an exact formula for the q-Laplace transform of $q^{x_n(t)}$.

Theorem 3.1.2. *Consider the q-Hahn AEP started from step initial data: $\forall n \in \mathbb{Z}_{>0}, x_n(0) = -n$. Then for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, we have the "Mellin-Barnes-type" Fredholm determinant formula*

$$\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n}; q\right)_\infty}\right] = \det\left(I + K_\zeta\right) \quad (3.2)$$

where $\det(I+K_\zeta)$ is the Fredholm determinant of $K_\zeta : L^2(C_1) \to L^2(C_1)$ for C_1 a positively oriented circle containing 1 with small enough radius so as to not contain 0, $1/q$ and $1/\nu$. The operator K_ζ is defined in terms of its integral kernel

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)} (-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds$$

with

$$g(w) = \left(\frac{(\nu w;q)_\infty}{(w;q)_\infty}\right)^n \exp\left((q-1)t\sum_{k=0}^{\infty} \frac{R}{\nu}\frac{\nu w q^k}{1-\nu w q^k} - L\frac{w q^k}{1-wq^k}\right)\frac{1}{(\nu w;q)_\infty}.$$

When the asymmetry parameters R and L of the q-Hahn AEP are set to $R = 1$ and $L = 0$, particles can jump only to the right. An application of the law of large numbers and the classical central limit theorem shows that there exist constants π and σ such that $x_1(t)/t$ converges almost surely to π and we have the convergence in distribution as t goes to infinity

$$\frac{x_1(t) - \pi t}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0, 1).$$

Such a result is true in particular for the TASEP, but what happens if one allows jumps to the left? Theorem 2 in [TW09] shows that for ASEP, that is if one allows *nearest-neighbour* jumps to the left, the position of the first particle still fluctuates on a $\sqrt{t}$ scale, but the limiting law is not Gaussian.

An asymptotic analysis of the Fredholm determinant in Theorem 3.1.2 when $\nu = q$ shows that the situation is very different when one allows *long-range* jumps to the left.

Theorem 3.1.3. *Consider the MADM exclusion process started from step initial condition. For asymmetry parameters R and $L = 1 - R$ such that $R_{min}(q) < R < 1$, where $R_{min}(q)$ is an explicit bound depending on the parameter q (see Theorem 3.5.4 and Remark 3.5.8 for a more precise statement), we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_1(t) - \pi t}{\sigma t^{1/3}} \geqslant x\right) = F_{\mathrm{GUE}}(-x),$$

where π and $\sigma > 0$ are explicit constants depending on R and q, and $F_{\mathrm{GUE}}(x)$ is the distribution function of the GUE Tracy-Widom distribution (see Definition 3.5.1).

Theorem 3.1.3 is proved in Section 3.5 as Theorem 3.5.4. The asymptotic analysis of the Fredholm determinant also allows for a similar result for particles in the bulk of the rarefaction fan. The following theorem about fluctuation of particles positions in the rarefaction fan is also proved in Section 3.5 as Theorem 3.5.2.

Theorem 3.1.4. *Consider the MADM exclusion process started from step initial condition, for asymmetry parameters R and $L = 1 - R$ such that $R > L \geqslant 0$. Assume that $\theta \in (0, +\infty)$ parametrizes the position in the rarefaction fan (see Section 3.4 and Theorem 3.5.2 for a more precise statement). There exists explicit constants $\kappa(\theta), \pi(\theta)$ and $\sigma(\theta)$, such that under the additional hypothesis $q^\theta > 2q/(1 + q)$, then for $n = \lfloor \kappa(\theta)t \rfloor$, we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_n(t) - \pi(\theta)t}{\sigma(\theta)t^{1/3}} \geqslant x\right) = F_{\mathrm{GUE}}(-x).$$

The expressions of the model-dependent constants $\kappa(\theta), \pi(\theta)$ and $\sigma(\theta)$ as functions of θ confirm the predictions of KPZ scaling theory (see Section 3.4).

Theorem 3.1.4 implies as a corollary a weak law of large numbers: for $n = \lfloor \kappa(\theta)t \rfloor$,

$$\frac{x_n(t)}{t} \xrightarrow[t\to\infty]{\mathbb{P}} \pi(\theta).$$

This law of large numbers implies a macroscopic density profile of the rarefaction fan as in Figure 3.3. The density profile in the partially asymmetric case (that is when $R > L > 0$) is discontinuous. Such a discontinuity of the macroscopic density profile has previously been exhibited in certain particle systems (e.g. [GKR10] studies a facilitated exclusion process

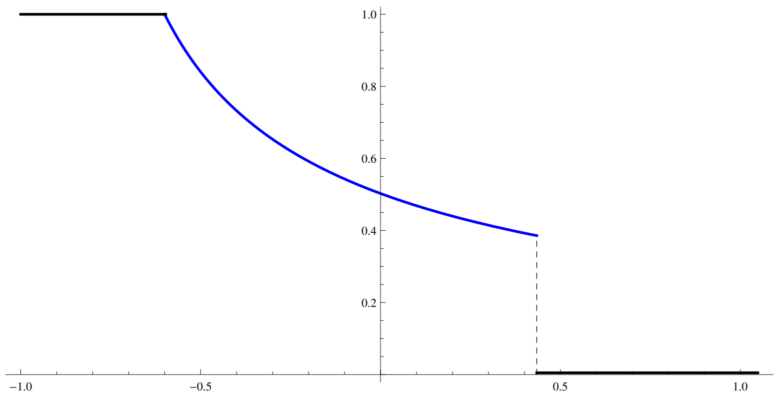

Figure 3.3: Density profile $x \mapsto \rho(x)$ for a q-Hahn AEP with $q = \nu = 0.6$, and asymmetry parameters $R = 0.8$ and $L = 0.2$, starting from step initial data. The density $\rho(x)$ has to be understood as the local density of particles at time t around site xt for very large t.

for which the density of particles stays above 1/2 when starting from step initial condition). However, to the authors knowledge, Theorem 3.1.3 provides the first limit theorem for the fluctuations of locations of particles at a downward (i.e. antishock) discontinuity of the density profile.

One can give a soft argument explaining why a discontinuity is present in the density profile. The rate at which the first particle jumps to the right is

$$\sum_{j=1}^{\infty} \phi^{R}_{q,\nu}(j|+\infty) < \infty.$$

The rate at which the first particle jumps to the left is

$$\sum_{j=1}^{m} \phi^{L}_{q,\nu}(j|m) \xrightarrow[m\to+\infty]{} +\infty,$$

where $m = x_1(t) - x_2(t) - 1$. Thus, even if particles have a drift to the right because $R > L$, the first particle stays with high probability at a bounded distance from the second particle, and hence the density around the first particles is strictly positive.

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Outline of the chapter

In Section 3.2.1, we provide definitions and establish useful identities for some q-deformed special functions that appear naturally in the next sections. In Section 3.3, we introduce the q-Hahn AEP and establish the Fredholm determinant identity of Theorem 3.1.2. In Section 3.4, we study this process from the point of view of the conjectural KPZ scaling theory, and we state the predicted limit theorems. We sketch an asymptotic analysis of the Fredholm determinant, leading to the predicted Tracy-Widom limit theorem. In Section 3.5, we make a rigorous asymptotic analysis in the case $\nu = q$, which corresponds to the MADM, thus proving Theorems 3.1.3 and 3.1.4.

3.2 Preliminaries on q-analogues and the q-Hahn distribution

3.2.1 Useful q-series

We first recall classical notations from the theory of q-analogues, that were already mentioned in 1.2.3. Fix hence forth that $q \in (0,1)$. For $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geqslant 0}$, define the q-Pochhammer symbol

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \qquad \text{and} \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

For an integer n, the q-integer $[n]_q$ is

$$[n]_q = 1 + q + \dots + q^{n-1} = \frac{1-q^n}{1-q}.$$

The q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q = \frac{(q;q)_n}{(1-q)^n}. \tag{3.3}$$

The q-binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

For $|z| < 1$, the q-binomial theorem [AAR99, Theorem 10.2.1] implies that

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.4}$$

The q-gamma function is defined by

$$\Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty},$$

and the q-digamma function is defined by

$$\Psi_q(z) = \frac{\partial}{\partial z}\log\Gamma_q(z).$$

From the definition of the q-digamma function, we have a series representation for Ψ_q,

$$\Psi_q(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma_q(z) = -\log(1-q) + \log(q)\sum_{k=0}^{\infty}\frac{q^{k+z}}{1-q^{k+z}}. \tag{3.5}$$

Let us also define a closely-related series that will appear in Section 3.4,

$$G_q(z) := \sum_{i=1}^{\infty} \frac{z^i}{[i]_q}.$$

Lemma 3.2.1. *For $z \in \mathbb{C}$ with positive real part,*

$$G_q(q^z) = \frac{1-q}{\log q}\big(\Psi_q(z) + \log(1-q)\big). \tag{3.6}$$

For $z \in \mathbb{C}$ with real part greater than -1,

$$G_{q^{-1}}(q^z) = \frac{q^{-1}-1}{\log q}\big(\Psi_q(z+1) + \log(1-q)\big). \tag{3.7}$$

Proof. Assume $z \in \mathbb{C}$ with positive real part. Using the series representation (3.5), we have that

$$\frac{1-q}{\log q}\big(\Psi_q(z) + \log(1-q)\big) = (1-q)\sum_{k=0}^{\infty} \frac{q^{k+z}}{1-q^{k+z}}.$$

Since z has positive real part, we can write for all $k \geqslant 0$

$$\frac{q^{k+z}}{1-q^{k+z}} = \sum_{i=1}^{\infty} q^{(k+z)i},$$

so that the right-hand-side in (3.6) equals

$$(1-q)\sum_{k=0}^{\infty}\sum_{i=1}^{\infty} q^{(k+z)i}.$$

Exchange the summations yields

$$\frac{1-q}{\log q}\big(\Psi_q(z) + \log(1-q)\big) = (1-q)\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} q^{(k+z)i} = \sum_{i=1}^{\infty} \frac{(q^z)^i}{[i]_q}.$$

Equation (3.7) can be deduced from (3.6) replacing z by $z+1$. □

A consequence of Lemma 3.2.1 is the following formula for the k-fold derivatives of the q-digamma function:

$$\Psi_q^{(k)}(z) = (\log q)^{k+1} \sum_{n=1}^{\infty} \frac{n^k q^{nz}}{1-q^n}. \tag{3.8}$$

3.2.2 A symmetry identity for the q-Hahn distribution

We first recall the definition of the q-Hahn distribution, already discussed in Section 1.2.

Definition 3.2.2. For $q \in (0,1)$, $0 \leqslant \nu \leqslant \mu < 1$ and integers $0 \leqslant j \leqslant m$, define the function

$$\varphi_{q,\mu,\nu}(j|m) = \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{m-j}}{(\nu;q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q,$$

where

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_j\,(q;q)_{m-j}}$$

are q-Binomial coefficients with, as usual,

$$(z;q)_n = \prod_{i=0}^{n-1} \left(1 - q^i z\right).$$

It happens that for each $m \in \mathbb{N} \cup \{\infty\}$, this defines a probability distribution on the set $\{0,\ldots,m\}$. The weights $\varphi_{q,\mu,\nu}(j|m)$ are very closely related to the weights associated with the q-Hahn orthogonal polynomials (see Digression 1.2.3 in Section 1.2).

Lemma 3.2.3 (Lemma 1.1, [Cor14])**.** *For any* $|q| < 1$ *and* $0 \leqslant \nu \leqslant \mu < 1$,

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1.$$

Proof. As shown in [Cor14], this equation is equivalent to a specialization of some known summation formula for basic hypergeometric series ${}_2\phi_1$ (Heine's q-generalizations of Gauss' summation formula). ☐

An interesting interpretation of the q-Hahn distribution is provided in Section 4 of [GO09]. The authors define a q-analogue of the Pólya urn process: One considers two urns, initially empty, in which one sequentially adds balls. When the first urn contains k balls, and the second urn contains $n-k$ balls, one adds a ball to the first urn with probability $[\tilde{\nu}-\tilde{\mu}+n-k]_q/[\tilde{\nu}+n]_q$, and we set $\mu = q^{\tilde{\mu}}$ and $\nu = q^{\tilde{\nu}}$. One adds a ball to the second urn with the complementary probability. Then $\varphi_{q,\mu,\nu}(j|m)$ is the probability that after m steps, the first urn contains j balls. When q goes to 1, one recovers the classical Pólya urn process.

For the classical Pólya urn, it is known that after n steps, the number of balls in the first urn is distributed according to the Beta-Binomial distribution. Further, the proportion of balls in the first urns converges in distribution to the Beta distribution when the number of added balls tends to infinity. Thus, it is natural to consider the q-Hahn distribution as a q-analogue of the Beta-Binomial distribution. Further, we expect that if X is a random variable drawn according to the q-Hahn distribution on $\{0,\ldots,m\}$ with parameters (q,μ,ν), the q-deformed proportion $[X]_q/[m]_q$ converges as m goes to infinity to a q analogue of the Beta distribution, which converges as q goes to 1 to the Beta distribution with parameters $(\tilde{\nu}-\tilde{\mu},\tilde{\mu})$.

It is known that the expectation of the Beta-Binomial distribution with parameters $(n,\tilde{\nu}-\tilde{\mu},\tilde{\mu})$ is

$$n\frac{\tilde{\nu}-\tilde{\mu}}{\tilde{\nu}},$$

that is a quantity proportional to n. Hence, if X (resp. Y) is a random variable following the Beta-Binomial distribution on $\{0,\dots,x\}$ (resp. $\{0,\dots,y\}$), with parameters $\tilde{\mu}$ and $\tilde{\nu}-\tilde{\mu}$, then we have

$$\mathbb{E}\left[xY\right] = \mathbb{E}\left[yX\right]. \tag{3.9}$$

It turns out that the (3.9) holds true also at the q-deformed level. The following proposition was discovered in [Cor14], but the proof that we present below is extracted from [Bar14]

Proposition 3.2.4 (Proposition 1.2, [Cor14])**.** *Fix $|q|<1$ and $0\leqslant \nu\leqslant \mu<1$. Let X (resp. Y) be a random variable following the q-Hahn distribution on $\{0,\dots,x\}$ (resp. $\{0,\dots,y\}$). We have*

$$\mathbb{E}\left[q^{xY}\right] = \mathbb{E}\left[q^{yX}\right].$$

In other terms,

$$\mathbb{E}\Big[[xY]_q\Big] = \mathbb{E}\Big[[yX]_q\Big].$$

Proof. Let $S_{x,y} := \sum_{j=0}^{x}\varphi_{q,\mu,\nu}(j|x)q^{jy}$. We have to show that $S_{x,y}=S_{y,x}$ for all integers $x,y\geqslant 0$. Our proof is based on the fact that $S_{x,y}$ satisfies a recurrence relation which is invariant when exchanging the roles of x and y. First notice that by Lemma 3.2.3, $S_{x,0}=1$ for all $x\geqslant 0$, and by definition $S_{0,y}=1$ for all $y\geqslant 0$.

The Pascal identity for q-Binomial coefficients, (see 10.0.3 in [AAR99]),

$$\begin{bmatrix} x+1 \\ j \end{bmatrix}_q = \begin{bmatrix} x \\ j \end{bmatrix}_q q^j + \begin{bmatrix} x \\ j-1 \end{bmatrix}_q,$$

yields

$$\begin{aligned} S_{x+1,y} &= \sum_{j=0}^{x+1}\mu^j\frac{(\nu/\mu;q)_j(\mu;q)_{x+1-j}}{(\nu;q)_{x+1}}\begin{bmatrix} x \\ j \end{bmatrix}_q q^jq^{jy} + \sum_{j=0}^{x+1}\mu^j\frac{(\nu/\mu;q)_j(\mu;q)_{x+1-j}}{(\nu;q)_{x+1}}\begin{bmatrix} x \\ j-1 \end{bmatrix}_q q^{jy}, \\ &= \sum_{j=0}^{x}\varphi_{q,\mu,\nu}(j|x)\frac{1-\mu q^{x-j}}{1-\nu q^x}q^jq^{jy} + \sum_{j=0}^{x}\varphi_{q,\mu,\nu}(j|x)\mu\frac{1-\nu/\mu q^j}{1-\nu q^x}q^yq^{jy}. \end{aligned}$$

The last equation can be rewritten

$$\begin{aligned} (1-\nu q^x)S_{x+1,y} &= \left(S_{x,y+1}-\mu q^x S_{x,y}\right) + \left(\mu q^y(S_{x,y}-\nu/\mu S_{x,y+1})\right), \\ &= (1-\nu q^y)S_{x,y+1} + \mu(q^y-q^x)S_{x,y}. \end{aligned}$$

Thus, the sequence $(S_{x,y})_{(x,y)\in\mathbb{N}^2}$ is completely determined by

$$\begin{cases} (1-\nu q^x)S_{x+1,y} = (1-\nu q^y)S_{x,y+1} + \mu(q^y-q^x)S_{x,y}, \\ S_{x,0}=S_{0,y}=1. \end{cases} \tag{3.10}$$

Setting $T_{x,y}=S_{y,x}$, one notices that the sequence $(T_{x,y})_{(x,y)\in\mathbb{N}^2}$ enjoys the same recurrence, which concludes the proof. $\square$

Remark 3.2.5. To completely avoid the use of basic hypergeometric series, one would also need a similar proof of the Lemma 3.2.3. One can prove the result by recurrence on m (as in the proof of [BC13, Lemma 1.3]), but the calculations are less elegant when $\nu\neq 0$.

More precisely, fix some m and suppose that for any $0 \leqslant \nu \leqslant \mu < 1$, $S_{m,0}(q,\mu,\nu) := \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1$. Pascal's identity yields

$$\begin{aligned} S_{m+1,0}(q,\mu,\nu) &= \frac{1-\mu}{1-\nu} S_{m,0}(q,q\mu,q\nu) + \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m)\mu \frac{1-\nu/\mu q^j}{1-\nu q^m}, \\ &= \frac{1-\mu}{1-\nu} S_{m,0}(q,q\mu,q\nu) + \frac{\mu}{1-\nu q^m}\left(S_{m,0}(q,\mu,\nu) - \nu/\mu S_{m,1}(q,\mu,\nu)\right). \end{aligned}$$

Then, using the recurrence formula (3.10) for $S_{m,1}(q,\mu,\nu)$, and applying the recurrence hypothesis, one obtains $S_{m+1,0}(q,\mu,\nu) = 1$.

Moreover, the formula for the expectation of the Beta-Binomial distribution also has its q-analogue.

Corollary 3.2.6. *Let X be a random variable following the q-Hahn distribution on $\{0,1,\dots,m\}$ with parameters (q,μ,ν). We have*

$$\mathbb{E}\Big[[X]_q\Big] = [m]_q \frac{\mu-\nu}{1-\nu}.$$

Proof. Direct consequence of Proposition 3.2.4. □

Remark 3.2.7. If $\mu = q^{\tilde{\mu}}$ and $\nu = q^{\tilde{\nu}}$, then the limit of

$$[m]_q \frac{\mu-\nu}{1-\nu}$$

as q goes to 1 is, as we should expect, the expectation of the Beta-Binomial distribution with parameters $(m, \tilde{\nu}-\tilde{\mu}, \tilde{\mu})$ i.e.

$$m\frac{\tilde{\nu}-\tilde{\mu}}{\tilde{\nu}}.$$

3.3 An asymmetric exclusion process solvable via Bethe ansatz

Let us recall the definition of the q-Hahn-TASEP [Pov13, Cor14]. Fix $q \in (0,1)$ and $0 \leqslant \nu < \mu < 1$. Then the N-particle q-Hahn TASEP is a discrete time Markov chain $\vec{x}(t) = \{x_n(t)\}_{n=0}^{N} \in \mathbb{X}^N$ where the state space $\mathbb{X}^N$ is

$$\mathbb{X}^N = \{+\infty = x_0 > x_1 > \cdots > x_N \; ; \; \forall n \geq 1, x_n \in \mathbb{Z}\}.$$

At time $t+1$, each coordinate $x_n(t)$ is updated independently and in parallel to $x_n(t+1) = x_n(t) + j_n$ where $0 \leqslant j_n \leqslant x_{n-1}(t) - x_n(t) - 1$ is drawn according to the q-Hahn probability distribution. The q-Hahn probability distribution on $j \in \{0,1,\dots,m\}$ is defined by

$$\varphi_{q,\mu,\nu}(j|m) = \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{m-j}}{(\nu;q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q. \tag{3.11}$$

The exact solvability of the q-Hahn TASEP comes from:

1. A Markov duality with a q-Hahn totally asymmetric zero-range process (q-Hahn TAZRP), which is essentially the same process, but described by the evolution of gaps between consecutive particles.

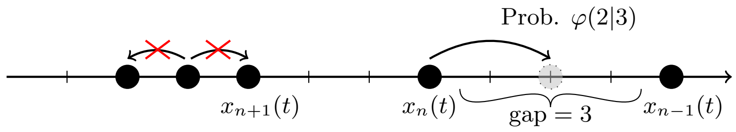

Figure 3.4: Jumps probabilities in the (discrete-time) q-Hahn TASEP.

2. The solvability of this q-Hahn zero-range process via the Bethe ansatz. Indeed the q-Hahn TAZRP was introduced by Povolotsky in [Pov13] as the most general parallel update discrete time totally asymmetric 'chipping' model on a ring lattice with factorized invariant measures which is solvable via Bethe ansatz.
3. The ability to express the q-Laplace transform as a Fredholm determinant using techniques introduced in the context of Macdonald processes [BC14].

In this section, we introduce a generalization of the q-Hahn TASEP allowing jumps in both directions such that the duality is preserved. More precisely, our construction generalizes a continuous-time degeneration of the q-Hahn TASEP by allowing jumps towards both directions. Proposition 1.2 in [Cor14], shows that certain 'q-moments' of the q-Hahn probability distribution enjoy a symmetry relation, which is ultimately responsible for an intertwining (and hence Markov duality) of the Markov generators of the q-Hahn Boson model and the q-Hahn TASEP:

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) q^{jy} = \sum_{j=0}^{y} \varphi_{q,\mu,\nu}(j|y) q^{jm}. \tag{3.12}$$

The same identity replacing all variables by their inverse also holds:

$$\sum_{j=0}^{m} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m) q^{-jy} = \sum_{j=0}^{y} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|y) q^{-jm}. \tag{3.13}$$

The weights $\varphi_{q,\mu,\nu}(j|m)$ and $\varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m)$ define probability distributions on $j \in \{0,1,\dots,m\}$. Notice also that

$$\varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m) = \left(\frac{\nu}{\mu}\right)^m \frac{1}{\nu^j} \varphi_{q,\mu,\nu}(j|m).$$

Notice that one can extend the q-Hahn weights by continuity when ν goes to zero. Thus,

$$\varphi_{q,\mu,0}(j|m) = \mu^j (\mu;q)_{m-j} \begin{bmatrix} m \\ j \end{bmatrix}_q \quad \text{and} \quad \varphi_{q^{-1},\mu^{-1},\infty}(j|m) = \mathbb{1}_{\{j=m\}}. \tag{3.14}$$

These observations motivate the introduction of a two-sided q-Hahn process where jumps to the left are distributed according to a q-Hahn distribution with parameters $q^{-1}, \mu^{-1}, \nu^{-1}$.

Definition 3.3.1. The discrete-time q-Hahn asymmetric zero-range process is a discrete-time Markov chain $\vec{y}(t) = \{y_i(t)\}_{i=0}^{\infty} \in \mathbb{Y}^{\infty}$, where

$$\mathbb{Y}^{\infty} = \left\{ (y_0, y_1, \dots) \,; \forall i \in \mathbb{Z}_{\geqslant 0},\ y_i \in \mathbb{Z}_{\geqslant 0} \text{ and } \sum_{i=0}^{\infty} y_i < \infty \right\}.$$

At time t, $y_i(t)$ particles are above site i for all $i \geqslant 0$. At time $t+1$, $\vec{y}(t)$ is updated to another state $\vec{y}(t+1)$ according to the following dynamics. For each site i, independently and in parallel, $s_i \in \{0, \dots, y_i(t)\}$ particles are transferred to site $i-1$ with probability $a \cdot \varphi_{q,\mu,\nu}\big(s_i|y_i(t)\big)$ or $t_i \in \{0, \dots, y_i(t)\}$ particles are transferred to site $i+1$ with probability $b \cdot \varphi_{q^{-1},\mu^{-1},\nu^{-1}}\big(t_i|y_i(t)\big)$. The variables a and b are asymmetry parameters such that $a+b = 1$, so that all of the above probabilities sum to 1. No particles are transferred out of site zero. Note that when $b = 0$, the q-Hahn AZRP reduces to the q-Hahn TAZRP from [Pov13].

One would like to define in the same manner a discrete-time q-Hahn asymmetric exclusion process generalizing the q-Hahn TASEP. However, it appears that it is not clear how to define such a process so that the duality between the discrete-time q-Hahn asymmetric zero-range process and the corresponding exclusion process is preserved. One of the main obstacles is that in order to satisfy the exclusion rule (particles must neither cross or occupy the same site), the positions of particles cannot be updated in parallel. This obstacle would of course vanish if particles could not jump simultaneously, which is generally the case for continuous time exclusion processes. A general procedure to build continuous time dynamics out of a discrete-time Markov chains is to scale the parameters so that the jump probabilities (here the coefficients $\varphi(j|y)$) are of order ϵ for all $j \geqslant 1$, and rescale time by setting $\tau := t\epsilon^{-1}$. As $\epsilon \to 0$, the process converges to a continuous time Markov process. We will see that by applying this procedure, one can find well-defined continuous time asymmetric q-Hahn exclusion and zero-range processes, and these processes are solvable in a similar way as it is done in [Cor14] for the q-Hahn TASEP and q-Hahn Boson.

3.3.1 General ν case

Let us fix $q, \nu \in (0,1)$ and set $\mu = \nu + (1-q)\epsilon$. Then for all $j \geqslant 1$, the jump probabilities of the discrete-time q-Hahn zero-range process from definition 3.3.1 become jump rates given by the limits,

$$a \cdot \varphi_{q,\mu,\nu}(j|m)/\epsilon \underset{\epsilon\to 0}{\longrightarrow} a\nu^{j-1}\left(\frac{1-q}{1-q^j}\right)\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}}, \tag{3.15}$$

$$b \cdot \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m)/\epsilon \underset{\epsilon\to 0}{\longrightarrow} b\nu^{-1}\left(\frac{1-q}{1-q^j}\right)\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}}. \tag{3.16}$$

Let us fix some notation and write these limiting rates as $\phi^R_{q,\nu}$ and $\phi^L_{q,\nu}$:

$$\phi^R_{q,\nu}(j|m) := R\frac{\nu^{j-1}}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}},$$
$$\phi^L_{q,\nu}(j|m) := L\frac{1}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}}.$$

The letters R and L stand for "right" and "left" as well as denote the values of the relative rates of jumps of particles in the process in those respective directions. Note that we deliberately removed the factor ν^{-1} (present in the $\epsilon \to 0$ limit) from $\phi^L_{q,\nu}(j|m)$ to be consistent with models previously introduced in the particle system literature (see Section 3.3.3). In this way, the rates are well-defined for $\nu = 0$ and all results of this section hold for $\nu = 0$ as well. It is useful for later calculations to notice that

$$R^{-1}\phi^R_{q^{-1},\nu^{-1}}(j|m) = \frac{\nu}{q}L^{-1}\phi^L_{q,\nu}(j|m). \tag{3.17}$$

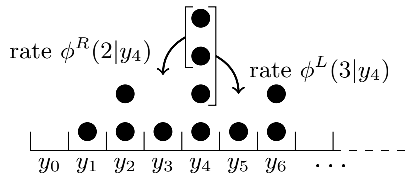

Figure 3.5: Rates of two possible transitions in the q-Hahn asymmetric zero-range process.

Definition 3.3.2. We define the (continuous time) q-Hahn asymmetric zero-range process (abbreviated q-Hahn AZRP) as a Markov process $\vec{y}(t) \in \mathbb{Y}^\infty$ with infinitesimal generator $B_{q,\nu}$ defined in (3.18). Before stating this generator, we must introduce some notation. For a vector $\vec{y} = (y_0, y_1, \dots)$, and any $j \leqslant y_i$ we denote

$$\vec{y}^{\,j}_{i,i-1} = (y_0, \dots, y_{i-1} + j, y_i - j, y_{i+1}, \dots),$$
$$\vec{y}^{\,j}_{i,i+1} = (y_0, \dots, y_{i-1}, y_i - j, y_{i+1} + j, \dots).$$

The operator $B_{q,\nu}$ is defined by its action on functions $\mathbb{Y}^\infty \to \mathbb{R}$ by

$$\big(B_{q,\nu} f\big)(\vec{y}) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{y_i} \phi^R_{q,\nu}(j|y_i) \Big(f(\vec{y}^{\,j}_{i,i-1}) - f(\vec{y}) \Big) + \sum_{j=1}^{y_i} \phi^L_{q,\nu}(j|y_i) \Big(f(\vec{y}^{\,j}_{i,i+1}) - f(\vec{y}) \Big) \right). \tag{3.18}$$

Informally, if the site i is occupied by y particles, j particles move together to site $i-1$ with rate $\phi^R_{q,\nu}(j|y)$ whereas j' particles move together to site $i+1$ with rate $\phi^L_{q,\nu}(j'|y)$, for all $1 \leqslant j, j' \leqslant y$ (see Figure 3.5).

Similarly, we define the continuous time asymmetric q-Hahn exclusion process as a Markov process $\vec{x}(t) \in \mathbb{X}^\infty$ where the state space $\mathbb{X}^\infty$ is defined by

$$\mathbb{X}^\infty = \left\{ +\infty = x_0 > x_1 > \dots > x_n > \dots \;\middle|\; \begin{array}{l} \forall n \geq 1,\ x_n \in \mathbb{Z} \\ \exists N > 0, \forall n \geqslant N, x_n - x_{n+1} = 1 \end{array} \right\}.$$

In words, $\mathbb{X}^\infty$ is the space of particle configurations that have a right-most particle and a left-most empty site. This is the analogue of the state space $\mathbb{Y}^\infty$ by exclusion/zero-range transformation, that is if one maps the gaps between consecutive particles in the exclusion process with the number of particles on the sites of the zero-range process.

The continuous time asymmetric q-Hahn exclusion process is defined by the action of its infinitesimal generator $T_{q,\nu}$. Let us introduce some notations. For a vector $\vec{x} = (x_0, x_1, \dots)$ we denote for any $j \in \mathbb{Z}$ and $i \geqslant 1$

$$\vec{x}^{\,j}_i = (x_0, \dots, x_{i-1}, x_i + j, x_{i+1}, \dots).$$

The operator $T_{q,\nu}$ acts on functions $\mathbb{X}^\infty \to \mathbb{R}$ by

$$\big(T_{q,\nu} f\big)(\vec{x}) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{x_{i-1}-x_i-1} \phi^R_{q,\nu}(j|x_{i-1} - x_i - 1) \Big(f(\vec{x}^{\,+j}_i) - f(\vec{x}) \Big) + \right.$$
$$\left. \sum_{j=1}^{x_i - x_{i+1}-1} \phi^L_{q,\nu}(j|x_i - x_{i+1} - 1) \Big(f(\vec{x}^{\,-j}_i) - f(\vec{x}) \Big) \right). \tag{3.19}$$

Remark 3.3.3. The q-Hahn AZRP (resp. q-Hahn AEP) might be defined on a larger state space including configurations with an infinite number of particles (resp. an infinite number of positive gaps between consecutive particles). Such a more general definition would add some complexity in several of the later statements. In the following, we study the zero-range processes only with a finite number of particles and the exclusion process starting only from the step-initial condition ($\forall n > 0, x_n(0) = -n$), thus we prefer to restrict our definition to the state-spaces $\mathbb{X}^\infty$ and $\mathbb{Y}^\infty$.

Before going further into the analysis of the q-Hahn AEP and AZRP, one should justify that they are well defined.

Existence of the q-Hahn AZRP Observe that the (finite) number of particles is conserved by the dynamics. Let k the number of particles in the initial condition. Then, each entry of the transition matrix of the process is bounded by

$$k \cdot \max_{m \in \{1,\dots,k\}} \sum_{j \leqslant m} \big(\phi^R_{q,\nu}(j|m) + \phi^L_{q,\nu}(j|m)\big) < \infty.$$

Then, the existence of a Markov process with the generator (3.18) follows from the classical construction of Markov chains on a denumerable state space with bounded generator (see e.g. [EK09, Chap. 4 Section 2]).

Existence of q-Hahn AEP Although it should be possible to show that the generator (3.19) defines uniquely a Markov semi-group (Using e.g. [BO12, Proposition 4.3]), we prefer to give a probabilistic construction of the q-Hahn AEP that corresponds to the generator. Fix some $T > 0$ and let us show that the processes is well-defined on the time interval $[0, T]$. Then, the construction will extend to any time $t \in \mathbb{R}_+$ by the Markov property. We prove that the construction on $[0, T]$ is actually that of a continuous-time Markov chain on a finite (random) state space. Consider a (possibly random) initial condition in $\mathbb{X}^\infty$. By the definition of the state space $\mathbb{X}^\infty$, there exists a (possibly random) integer N such that for all $n > N$, $x_n(0) - x_{n+1}(0) = 1$. We claim that almost surely, there exists an integer $n > N$ such that the particle labelled by n does not move on the time interval $[0, T]$. Indeed, if this particle moves, then it has to move at least once to its right, since there is no room to its left. The rates at which a jump on the right occurs is bounded by

$$M := \sup_{m \geqslant 1} \sum_{j=1}^{m} \phi^R_{q,\nu}(j|m) < \infty.$$

Since all particles are equipped with independent Poisson clocks, there exists almost surely a particle that does not jump to the right. Finally, the q-Hahn AEP can be constructed on $[0, T]$ as a Markov chain on a finite state-space.

We come now to the duality between the q-Hahn AEP and the q-Hahn AZRP.

Proposition 3.3.4. *Define $H : \mathbb{X}^\infty \times \mathbb{Y}^\infty \to \mathbb{R}$ as*

$$H(\vec{x}, \vec{y}) := \prod_{i=0}^{\infty} q^{(x_i+i)y_i}, \tag{3.20}$$

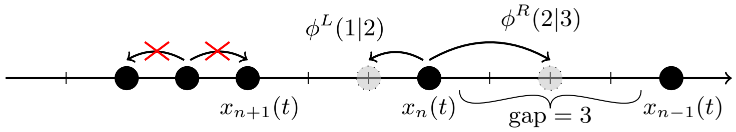

Figure 3.6: Rates of two possible jumps in the q-Hahn asymmetric exclusion process.

with the convention that the product is 0 *when* $y_0 > 0$. *For any* $(\vec{x}, \vec{y})$ *in* $\mathbb{X}^\infty \times \mathbb{Y}^\infty$, *we have that*

$$B_{q,\nu} H(\vec{x}, \vec{y}) = T_{q,\nu} H(\vec{x}, \vec{y}),$$

where $B_{q,\nu}$ *acts on the* $\vec{y}$ *variable,* $T_{q,\nu}$ *acts on the* $\vec{x}$ *variable.*

Proof. Under the scalings above and when ϵ goes to zero, identities (3.12) and (3.13) degenerate to

$$\sum_{j=1}^{m} \phi^R_{q,\nu}(j|m) \left(q^{jy} - 1\right) = \sum_{j=1}^{y} \phi^R_{q,\nu}(j|y) \left(q^{jm} - 1\right), \tag{3.21}$$

and

$$\sum_{j=1}^{m} \phi^L_{q,\nu}(j|m) \left(q^{-jy} - 1\right) = \sum_{j=1}^{y} \phi^L_{q,\nu}(j|y) \left(q^{-jm} - 1\right). \tag{3.22}$$

Let us explain how (3.21) is obtained. From the limit (3.15), we know that for $j \geqslant 1$,

$$\varphi_{q,\mu,\nu}(j|m) = \epsilon R^{-1} \phi^R_{q,\nu}(j|m) + o(\epsilon).$$

Since $\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1$, we know that

$$\varphi_{q,\mu,\nu}(0|m) = 1 - \sum_{j=1}^{m} \epsilon R^{-1} \phi^R_{q,\nu}(j|m) + o(\epsilon).$$

Finally, in terms of ϵ, identity (3.12) writes

$$1 - \sum_{j=1}^{m} \epsilon R^{-1} \phi^R_{q,\nu}(j|m) + \sum_{j=1}^{m} \epsilon R^{-1} \phi^R_{q,\nu}(j|m) q^{jy} + o(\epsilon) =$$
$$1 - \sum_{j=1}^{y} \epsilon R^{-1} \phi^R_{q,\nu}(j|y) + \sum_{j=1}^{y} \epsilon R^{-1} \phi^R_{q,\nu}(j|y) q^{jm} + o(\epsilon).$$

Substracting 1 from both sides and keeping only terms of order ϵ, one gets identity (3.21). Identity (3.22) is obtained in a similar way.

Applying generators $B_{q,\nu}$ and $T_{q,\nu}$ to the function $H(\vec{x}, \vec{y}) = \prod_{i=0}^{\infty} q^{(x_i+i)y_i}$ and using Equations (3.21) and (3.22) to each term of the sum, one gets that $B_{q,\nu} H = T_{q,\nu} H$. Let us write this more precisely.

$$T_{q,\nu}H(\vec{x},\vec{y}) = \prod_{i=1}^{\infty}\left(\sum_{j_i=0}^{x_{i-1}-x_i-1}\phi^R_{q,\nu}(j_i|x_{i-1}-x_i-1)\big(q^{j_i y_i}-1\big) \;+\right.$$
$$\left.\sum_{k_i=0}^{x_i-x_{i+1}-1}\phi^L_{q,\nu}(k_i|x_i-x_{i+1}-1)\big(q^{-k_i y_i}-1\big)\right)\prod_{i=0}^{\infty}q^{(x_i+i)y_i}.$$

Applying Equations (3.21) and (3.22) to the terms inside the parenthesis, we find that

$$\begin{aligned} T_{q,\nu}H(\vec{x},\vec{y}) &= \prod_{i=1}^{\infty}\left(\sum_{s_i=0}^{y_i}\phi^R_{q,\nu}(s_i|y_i)\big(q^{s_i(x_{i-1}-x_i-1)}-1\big) \;+\right. \\ &\quad \left.\sum_{t_i=0}^{y_i}\phi^L_{q,\nu}(t_i|y_i)\big(q^{-t_i(x_i-x_{i+1}-1)}-1\big)\right)\prod_{i=0}^{\infty}q^{(x_i+i)y_i} \\ &= B_{q,\nu}H(\vec{x},\vec{y}). \end{aligned}$$

□

Remark 3.3.5. One can see from the proof of Proposition 3.3.4 that our statement could be generalized:

- The duality still holds when the parameter ν is not the same for the jumps to the left and the jumps to the right.
- The parameter ν and the asymmetry parameters R and L could also depend on site/particle provided that the parameters corresponding to the i^{th} particle in the exclusion process equal the parameters corresponding to the i^{th} site in the zero-range process.

It is not presently clear if the solvabilty of the q-Hahn AZRP (resp. q-Hahn AEP) process extends to the general time and site-dependent (resp. particle-dependent) parameters beyond duality, see [Cor14, Section 2.4] for a related discussion in the q-Hahn TASEP case.

The k-particle q-Hahn AZRP process can be alternatively described in terms of ordered particle locations $\vec{n}(t) = \vec{n}(\vec{y}(t))$. The bijection between $\vec{n}$ coordinates and $\vec{y}$ coordinates is such that $n_i(t) = n$ if and only if $\sum_{j>n} y_j < i \leqslant \sum_{j\geqslant n} y_j$ and we impose that $\vec{n} \in \mathbb{W}^k$ where the Weyl chamber $\mathbb{W}^k$ is defined as

$$\mathbb{W}^k = \{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \;;\; n_i \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant i \leqslant k\}\,. \tag{3.23}$$

For a subset $I \subset \{1,\dots,k\}$ and $\vec{n} \in \mathbb{W}^k$, we introduce the vector $\vec{n}_I^+$ obtained from $\vec{n}$ by increasing by one all coordinates with index in I ; and the vector $\vec{n}_I^-$ obtained from $\vec{n}$ by decreasing by one all coordinates with index in I. As an example,

$$\vec{n}_i^+ = (n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_k).$$

With a slight abuse of notations, we will use the same symbol $B_{q,\nu}$ for the generator of the q-Hahn AZRP described in terms of variables in either $\mathbb{Y}^\infty$ or $\mathbb{W}^k$.

Definition 3.3.6. We say that $h : \mathbb{R}_+ \times \mathbb{W}^k$ solves the k-particle true evolution equation with initial data h_0 if it satisfies the conditions that:

1. for all $\vec{n} \in \mathbb{W}^k$ and $t \in \mathbb{R}_+$,
$$\frac{\mathrm{d}}{\mathrm{d}t} h(t, \vec{n}) = B_{q,\nu} h(t, \vec{n}),$$
2. for all $\vec{n} \in \mathbb{W}^k$, $h(t, \vec{n}) \xrightarrow[t \to 0]{} h_0(\vec{n})$,
3. for any $T > 0$, there exists constants $c, C > 0$ such that for all $\vec{n} \in \mathbb{W}^k$, $t \in [0, T]$,
$$|h(t, \vec{n})| \leqslant Ce^{c\|\vec{n}\|},$$
and for all $\vec{n}, \vec{n}' \in \mathbb{W}^k$, $t \in [0, T]$,
$$|h(t, \vec{n}) - h(t, \vec{n}')| \leqslant C|e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|,$$
where we define the norm of a vector in $\mathbb{W}^k$ by $\|\vec{n}\| = \sum_{i=1}^k n_i$.

Proposition 3.3.7. *Consider any initial data h_0 such that there exists constants $c, C > 0$ such that for all $\vec{n} \in \mathbb{W}^k$, $|h_0(\vec{n})| \leqslant Ce^{c\|\vec{n}\|}$, and for all $\vec{n}, \vec{n}' \in \mathbb{W}^k$, $|h_0(\vec{n}) - h_0(\vec{n}')| \leqslant C|e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|$. Then the solution of the true evolution equation is unique.*

Proof. We provide a probabilistic proof adapted from [BCS14, Appendix C]. Given $\vec{n}(t)$, a q-Hahn AZRP started from initial condition $\vec{n}(0) = \vec{n}$, we use a representation of any solution to the true evolution equation as a functional of the q-Hahn AZRP.

Let h^1 and h^2 two solutions of the true evolution equation with initial data h_0. Then $g := h^1 - h^2$ solves the true evolution equation with zero initial data. Let $T > 0$. Our aim is to prove that for any $\vec{n} \in \mathbb{W}^k$, $g(T, \vec{n}) = 0$. The idea is the following: By *formally* differentiating the function $t \mapsto \mathbb{E}^{\vec{n}}[g(t, \vec{n}(T-t))]$ we find a zero derivative. Thus we expect that this function is constant, and hence its value for $t = T$, which is $g(T, \vec{n})$, equals the limit when t goes to zero, which is expected to be 0. Of course, these formal manipulations need to be justified and we will see how condition (3) of the true evolution equation applies.

By condition (3) of the true evolution equation, there exist constants $c, C > 0$ such that for $t \in [0, T]$,
$$|g(t, \vec{n})| \leqslant Ce^{c\|\vec{n}\|}. \tag{3.24}$$

Let us first prove that on $[0, T]$, $\|\vec{n}(t)\| - \|\vec{n}\|$ can be bounded by a Poisson random variable N_T. Indeed, we have that for any $0 \leqslant t \leqslant T$,
$$\mathbb{P}^{\vec{n}}\left(\|\vec{n}(t)\| - \|\vec{n}\| = N\right) \leqslant \mathbb{P}\left(\text{ at least } \frac{N}{k} \text{ events on the right occurred on } [0, T]\right).$$

The rate of an event on the right is crudely bounded by $k\lambda$ where $\lambda = \max_{j \leqslant m \leqslant k} \phi^L(j|m) < \infty$. Thus, $\|\vec{n}(t)\| - \|\vec{n}\|$ can be bounded by a Poisson random variable N_T depending only on the horizon time T.

Consider the function $[0, T] \to \mathbb{R}$, $t \mapsto \mathbb{E}^{\vec{n}}[g(t, \vec{n}(T-t))]$. Given the exponential bound (3.24) and the inequality $\|\vec{n}(t)\| \leqslant \|\vec{n}\| + N_T$, this function is well-defined. Moreover, one can apply dominated convergence to show that it is continuous. Thus, the limit when t goes to zero is zero (because of the initial condition for g).

Let us show that the function is constant. First, observe that for $t \in [0, T]$,
$$B_{q,\nu} g(t, \vec{n}) \leqslant \sum_{\vec{n} \to \vec{n}'} 2k\lambda |g(t, \vec{n}')| \leqslant (2k)^2 \lambda C e^{c(\|\vec{n}\| + k)}.$$

Since $\vec{n}(T-t)$ can be bounded by $\|\vec{n}\| + N_T$,

$$|B_{q,\nu} g(t, \vec{n}(T-t))| \leqslant (2k)^2 \lambda C e^{c(\|\vec{n}\|+k+N_T)}. \tag{3.25}$$

Consider the function $\phi : [0,T]^2 \to \mathbb{R}$ defined by $\phi(t,s) = \mathbb{E}^{\vec{n}}[g(t, \vec{n}(s))]$. Since the right-hand-side of (3.25) is integrable, one can take the partial derivative of ϕ with respect to t inside the expectation, and we get

$$\frac{\partial \phi}{\partial t}(t,s) = \mathbb{E}^{\vec{n}}[B_{q,\nu} g(t, \vec{n}(s))]$$

The equality comes from condition (1) of true evolution equation, using dominated convergence. By condition (3) of the true evolution equation, we also have that for $t \in [0,T]$,

$$|g(t,\vec{n}) - g(t,\vec{n}')| \leqslant C|e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|. \tag{3.26}$$

Hence, for any fixed $t \in [0,T]$, we have for $0 < s < s' < T$

$$\left|\frac{\phi(t,s') - \phi(t,s)}{s'-s}\right| \leqslant C\mathbb{E}^{\vec{n}}\left[\frac{|e^{c\|\vec{n}(s')\|} - e^{c\|\vec{n}(s)\|}|}{s-s'}\right]. \tag{3.27}$$

Since one can bound $|\|\vec{n}(s)\| - \|\vec{n}(s')\||$ by a Poisson random variable with parameter proportional to $s'-s$, the right-hand-side of (3.27) has a limit when s' goes to s. This means that for any $t \in [0,T]$, the function $\vec{n} \mapsto g(t,\vec{n})$ is in the domain of the semi-group (of the q-Hahn AZRP). Thus, applying Kolmogorov backward equation and using the commutativity of the generator with the semi-group, we have that

$$\frac{\partial \phi}{\partial s}(t,s) = \mathbb{E}^{\vec{n}}[B_{q,\nu} g(t, \vec{n}(s))].$$

Consequently the derivative of $t \mapsto \mathbb{E}^{\vec{n}}[g(t, \vec{n}(T-t))]$ is zero. Hence the function is constant, and the value at $t = T$, $g(T,\vec{n})$ equals the limit when $t \to 0$ which is zero. □

Corollary 3.3.8. *For any fixed $\vec{x} \in \mathbb{X}^\infty$, the function $u : \mathbb{R}_+ \times \mathbb{W}^k \to \mathbb{R}$ defined by*

$$u(t,\vec{n}) = \mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{n})]$$

satisfies the true evolution equation with initial data $h_0(\vec{n}) = H(\vec{x}, \vec{n})$. As a consequence, the q-Hahn AEP and the k-particle q-Hahn AZRP are dual with respect to the function H, that is for any $\vec{x} \in \mathbb{X}^\infty$ and $\vec{n} \in \mathbb{W}^k$,

$$\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{n})] = \mathbb{E}^{\vec{n}}[H(\vec{x}, \vec{n}(t))].$$

Proof. By the Kolmogorov backward equation for the q-Hahn AZRP, it is clear that $(t,\vec{n}) \mapsto \mathbb{E}^{\vec{n}}[H(\vec{x}, \vec{n}(t))]$ satisfies the true evolution equation with initial data $\mathbb{E}[H(\vec{x}, \vec{n})]$ (the growth condition is clear). On the other hand, Kolmogorov backward equation for the q-Hahn AEP yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{n})] = T_{q,\nu}\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{n})] = \mathbb{E}^{\vec{x}}[T_{q,\nu} H(\vec{x}(t), \vec{n})].$$

Proposition 3.3.4 then implies

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,\vec{n}) = \mathbb{E}^{\vec{x}}[B_{q,\nu} H(\vec{x}(t), \vec{n})] = B_{q,\nu} u(t,\vec{n}).$$

Since u satisfies the growth condition and the initial condition, u solves the true evolution equation. Hence, by Proposition 3.3.7, we have that for all $\vec{x} \in \mathbb{X}^\infty$ and $\vec{n} \in \mathbb{W}^k$,

$$\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{n})] = \mathbb{E}^{\vec{n}}[H(\vec{x}, \vec{n}(t))].$$

□

In order to compute the observables $\mathbb{E}\left[\prod_{i=1}^k q^{x_{n_i}(t)+n_i}\right]$, it would be natural to solve the true evolution equation. However, it is not clear how to proceed directly, and Proposition 3.3.9 provides an important reduction by rewriting the k-particle true evolution equation as a k-particle free evolution equation with $k-1$ two-body boundary conditions.

Proposition 3.3.9. *Let $\vec{x}(\cdot)$ denote the q-Hahn AEP. If $u : \mathbb{R}_+ \times \mathbb{Z}^k \to \mathbb{C}$ solves:*

1. (k-particle free evolution equation) *for all $\vec{n} \in \mathbb{Z}^k$ and $t \in \mathbb{R}_+$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t; \vec{n}) = \frac{1-q}{1-\nu} \sum_{i=1}^{k} \left[R\left(u(t; \vec{n}_i^-) - u(t; \vec{n})\right) + L\left(u(t; \vec{n}_i^+) - u(t; \vec{n})\right) \right];$$

2. ($k-1$ two-body boundary conditions) *for all $\vec{n} \in \mathbb{Z}^k$ and $t \in \mathbb{R}_+$ if $n_i = n_{i+1}$ for some $i \in \{1, \dots, k-1\}$ then*

$$\alpha u(t; \vec{n}_{i,i+1}^-) + \beta u(t; \vec{n}_{i+1}^-) + \gamma u(t; \vec{n}) - u(t; \vec{n}_i^-) = 0$$

where the parameters α, β, γ are defined in terms of q and ν as

$$\alpha = \frac{\nu(1-q)}{1-q\nu}, \qquad \beta = \frac{q-\nu}{1-q\nu}, \qquad \gamma = \frac{1-q}{1-q\nu};$$

3. (initial data) *for all $\vec{n} \in W_k$, $u(0; \vec{n}) = \mathbb{E}\left[\prod_{i=1}^k q^{x_{n_i}(0)+n_i}\right]$;*
4. *for any $T > 0$, there exists constants $c, C > 0$ such that for all $\vec{n} \in W_k$, $t \in [0, T]$,*

$$|u(t; \vec{n})| \leqslant C e^{c\|\vec{n}\|},$$

and for all $\vec{n}, \vec{n}' \in \mathbb{W}^k$, $t \in [0, T]$,

$$|h(t, \vec{n}) - h(t, \vec{n}')| \leqslant C |e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|;$$

then for all $\vec{n} \in W_k$ and all $t \in \mathbb{R}_+$, $u(t; \vec{n}) = \mathbb{E}\left[\prod_{i=1}^k q^{x_{n_i}(t)+n_i}\right]$.

Proof. In the totally asymmetric case, that is when $R = 1$ and $L = 0$, this result can be seen as a degeneration of Proposition 1.7 in [Cor14].

First we show that conditions (1) and (2) imply that u satisfies condition (1) of the true evolution equation in Definition 3.3.6. Condition (2) in Proposition 3.3.9 says that for all $\vec{n}$ such that $n_i = n_{i+1}$,

$$\frac{\nu(1-q)}{1-q\nu} u(t; \vec{n}_{i,i+1}^-) + \frac{q-\nu}{1-q\nu} u(t; \vec{n}_{i+1}^-) + \frac{1-q}{1-q\nu} u(t; \vec{n}) - u(t; \vec{n}_i^-) = 0. \tag{3.28}$$

This is equivalent to saying that for all $\vec{n}$ such that $n_i = n_{i+1}$,

$$\frac{\nu^{-1}(1-q^{-1})}{1-q^{-1}\nu^{-1}} u(t; \vec{n}_{i,i+1}^+) + \frac{q^{-1}-\nu^{-1}}{1-q^{-1}\nu^{-1}} u(t; \vec{n}_i^+) + \frac{1-q^{-1}}{1-q^{-1}\nu^{-1}} u(t; \vec{n}) - u(t; \vec{n}_{i+1}^+) = 0. \tag{3.29}$$

Indeed, if we set $\vec{m} := \vec{n}^{-}_{i,i+1}$ in (3.28), we have that $\vec{n}^{-}_{i+1} = \vec{m}^{+}_{i}, \vec{n} = \vec{m}^{+}_{i,i+1}$ and $\vec{n}^{-}_{i} = \vec{m}^{+}_{i+1}$. Dividing the numerator and the denominator of each coefficient in (3.28) by $-q\nu$, we have

$$\begin{aligned}\frac{\nu(1-q)}{1-q\nu}u(t;\vec{n}^{-}_{i,i+1}) &= \frac{1-q^{-1}}{1-q^{-1}\nu^{-1}}u(t;\vec{m}),\\ \frac{q-\nu}{1-q\nu}u(t;\vec{n}^{-}_{i+1}) &= \frac{q^{-1}-\nu^{-1}}{1-q^{-1}\nu^{-1}}u(t;\vec{m}^{+}_{i}),\\ \frac{1-q}{1-q\nu}u(t;\vec{n}) &= \frac{\nu^{-1}(1-q^{-1})}{1-q^{-1}\nu^{-1}}u(t;\vec{m}^{+}_{i,i+1}).\end{aligned}$$

Finally we get exactly (3.29) with $\vec{n}$ replaced by $\vec{m}$.

The next lemma explains the effect of the boundary condition.

Lemma 3.3.10. *Suppose that a function $f : \mathbb{Z}^m \to \mathbb{R}$ satisfies the boundary conditions that for all $\vec{n}$ such that $n_i = n_{i+1}$ for some $i \in \{1, \ldots, k-1\}$,*

$$\alpha f(\vec{n}^{-}_{i,i+1}) + \beta f(\vec{n}^{-}_{i+1}) + \gamma f(\vec{n}) - f(\vec{n}^{-}_{i}) = 0.$$

Then for $\vec{n} = (n, \ldots, n)$, the function f also satisfies

$$\sum_{i=1}^{m} R\frac{1-q}{1-\nu}\left(f(\vec{n}^{-}_{i}) - f(\vec{n})\right) = \sum_{j=1}^{m}\phi^{R}_{q,\nu}(j|m)f(\underbrace{n,\ldots,n}_{m-j},\underbrace{n-1,\ldots,n-1}_{j}), \tag{3.30}$$

and

$$\sum_{i=1}^{m} L\frac{1-q}{1-\nu}\left(f(\vec{n}^{+}_{i}) - f(\vec{n})\right) = \sum_{j=1}^{m}\phi^{L}_{q,\nu}(j|m)f(\underbrace{n+1,\ldots,n+1}_{j},\underbrace{n,\ldots,n}_{m-j}). \tag{3.31}$$

Proof. Equation (3.30) is exactly the conclusion of Lemma 2.4 in [Cor14] with $\mu = \nu + (1-q)\epsilon$ and keeping only the terms of order ϵ. For completeness, we will give a direct proof as well. Theorem 1 in [Pov13] states that an associative algebra generated by A, B obeying the quadratic homogeneous relation

$$BA = \alpha AA + \beta AB + \gamma BB, \tag{3.32}$$

enjoys the following non-commutative analogue of Newton binomial expansion

$$\left(\frac{\mu-\nu}{1-\nu}A + \frac{1-\mu}{1-\nu}B\right)^m = \sum_{j=0}^{m}\varphi_{q,\mu,\nu}(j|m)A^jB^{m-j}.$$

Let $\mu = \nu+(1-q)\epsilon$ and consider only the terms of order ϵ as $\epsilon \to 0$ in the above expression. By identification of $O(\epsilon)$ terms, we have

$$\sum_{i=1}^{m}\frac{1-q}{1-\nu}B^{i-1}AB^{m-i} = \sum_{j=1}^{m}R^{-1}\phi^{R}_{q,\nu}(j|m)A^jB^{m-j}. \tag{3.33}$$

Interpreting each monomial of the form $X_1X_2\ldots X_m$ with $X_i \in \{A, B\}$ as $f(n_1, \ldots, n_m)$ where $n_i = n$ if $X_i = B$ and $n_i = n-1$ if $X_i = A$, the boundary condition in the statement of the Lemma corresponds algebraically to the quadratic relation (3.32). Thus we find that for $\vec{n} = (n, \dots, n)$, f satisfies

$$\sum_{i=1}^{m} R\frac{1-q}{1-\nu}\left(f(\vec{n}_i^-) - f(\vec{n})\right) = \sum_{j=1}^{m} \phi^R_{q,\nu}(j|m) f(\underbrace{n, \dots, n}_{m-j}, \underbrace{n-1, \dots, n-1}_{j}).$$

Since (3.33) is true as an identity in an algebra over the field of rational fractions in q and ν, we can certainly replace q and ν by their inverses. Keeping in mind (3.17), we find that

$$\sum_{i=1}^{m} \frac{1-q^{-1}}{1-\nu^{-1}} B^{i-1}AB^{m-i} = \frac{\nu}{q}\sum_{j=1}^{m} L^{-1}\phi^L_{q,\nu}(j|m) A^j B^{m-j}. \tag{3.34}$$

Interpreting the monomials as $f(n_1, \dots, n_m)$ with $n_i = n$ or $n+1$, we get that

$$\sum_{i=1}^{m} L\frac{1-q}{1-\nu}\left(f(\vec{n}_i^+) - f(\vec{n})\right) = \sum_{j=1}^{m} \phi^L_{q,\nu}(j|m) f(\underbrace{n+1, \dots, n+1}_{j}, \underbrace{n, \dots, n}_{m-j}).$$

□

The application of Lemma 3.30 for each cluster of equal elements in $\vec{n}$ shows that under conditions (1) and (2), $u(t;\vec{n})$ satisfies condition (1) of Definition 3.3.6

$$\frac{\mathrm{d}}{\mathrm{d}t} h(t,\vec{n}) = B_{q,\nu} h(t,\vec{n}).$$

The growth condition (3) of the true evolution equation is exactly the same as condition (4) of the Proposition with the same constants c, C, and the initial data are the same. Hence, if u satisfies the conditions of the Proposition, it solves the true evolution equation with initial data $h_0(\vec{y}) = H(\vec{x},\vec{y})$, and by Proposition 3.3.7, $u(t;\vec{n}) = \mathbb{E}\left[\prod_{i=1}^{\infty} q^{x_{n_i}(t)+n_i}\right]$. □

Remark 3.3.11. In the case $\nu = q$, the system of ODEs with two-body boundary conditions in Proposition 3.3.9 was already known, see (10) and (12) in [SW98b].

Proposition 3.3.12 provides an exact contour integral formula for the observables

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right].$$

We simply check that the formula is a solution to the true evolution equation, using Proposition 3.3.9. The form of this formula originates in the theory of Macdonald processes [BC14] and similar formulas have been obtained as solutions of true evolution equations in [BCS14] and subsequent papers.

Proposition 3.3.12. *Fix $q \in (0,1)$, $0 \leqslant \nu < 1$, and integer k. Consider the continuous time q-Hahn exclusion process started from step initial data (i.e. $x_n(0) = -n$ for $n \geq 1$). Then for any $\vec{n} \in \mathbb{W}^k$,*

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - qz_B}$$
$$\prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \exp\left((q-1)t\left(\frac{Rz_j}{1-\nu z_j} - \frac{Lz_j}{1-z_j}\right)\right) \frac{dz_j}{z_j(1-\nu z_j)}. \tag{3.35}$$

where the integration contours $\gamma_1, \dots, \gamma_k$ *are chosen so that they all contain* 1, γ_A *contains* $q\gamma_B$ *for* $B > A$ *and all contours exclude* 0 *and* $1/\nu$.

Proof. We prove that the right-hand-side of (3.35) verifies the conditions of Proposition 3.3.9. Note that (3.35) is very similar with the result of Theorem 1.9 in [Cor14] for q-Hahn TASEP, the only difference being that the factor $((1-\mu z_j)/(1-\nu z_j))^t$ is replaced by

$$\exp\left((q-1)t\left(\frac{Rz_j}{1-\nu z_j} - \frac{Lz_j}{1-z_j}\right)\right).$$

Let us explain briefly why conditions (2) and (3) are verified: As it is explained in the proof of Theorem 1.9 in [Cor14], the application of the boundary condition to the integrand brings out an additional factor

$$\frac{(1-\nu)^2}{(1-q\nu)(1-\nu z_i)(1-\nu z_{i+1})}(z_i - qz_{i+1}).$$

The factor $(z_i - qz_{i+1})$ cancels out the pole separating the contours for the variables z_i and z_{i+1}. We may then take the same contour and use antisymmetry to prove that the integral is zero. To check the initial data, one may observe by residue calculus that the integral is zero when $n_k \le 0$ since there is no pole at 1 for the z_k integral; and one verifies that the integral equals 1 in the alternative case by sending the contours to infinity (this is the same calculation as in [Cor14]).

Let us check the free evolution equation. The generator of the free evolution equation can be written as a sum $\sum_{i=1}^k \mathcal{L}_i$ where $\mathcal{L}_i$ acts by

$$\mathcal{L}_i f = \frac{1-q}{1-\nu}\left[R\left(f(\vec{n}_i^-) - f(\vec{n})\right) + L\left(f(\vec{n}_i^-) - f(\vec{n})\right)\right].$$

Applying $\mathcal{L}_i$ to the R.H.S of (3.35) brings inside the integration a factor

$$\frac{1-q}{1-\nu}\left(R\left(\frac{1-z_i}{1-\nu z_i} - 1\right) + L\left(\frac{1-\nu z_i}{1-z_i} - 1\right)\right)$$

which is readily shown to equal the argument of the exponential.

Finally, let us check the growth condition. Let us denote by $\tilde{u}(t, \vec{n})$ the right-hand-side of (3.35). One can choose the contours $\gamma_1, \dots, \gamma_k$ such that for all $1 \leqslant A < B \leqslant k$ and $1 \leqslant j \leqslant k$, $|z_A - qz_B|$, $|1 - z_j|$, $|1 - \nu z_j|$ and $|z_j|$ are uniformly bounded away from zero. Since the contours are finite, one can find constants c_1, c_2 and c_3, such that for any t smaller that some arbitrary but fixed constant T,

$$|\tilde{u}(t, \vec{n})| \leqslant c_1 \prod_{j=1}^{k} \left(c_2^{n_j} \exp((1-q)tc_3)\right),$$

and

$$|\tilde{u}(t, \vec{n}) - \tilde{u}(t, \vec{n}')| \leqslant c_1 \exp(k(1-q)tc_3)|c_2^{\|\vec{n}\|} - c_2^{\|\vec{n}'\|}|,$$

where c_1, c_2 and c_3 depend only on the parameters q, ν, the choice of contours and the horizon time T. $\square$

Proposition 3.3.12 provides a formula for all integer moments of the random variable $q^{x_n(t)+n}$ when the continuous time two-sided q-Hahn exclusion process is started from step initial condition. Since $q \in (0,1)$ and $x_n(t)+n \geq 0$, this completely characterizes the law of $x_n(t)$. In order to extract information out of these expressions, we give a Fredholm determinant formula for the q-Laplace transform of $q^{x_n(t)+n}$, following an approach designed initially for the study of Macdonald processes [BC14]. The reader is referred to [BC14, Section 3.22] for some background on Fredholm determinants. In the totally asymmetric case ($L=0$), Theorem 3.3.13 can also be seen as a degeneration when ϵ goes to zero of Theorem 1.10 in [Cor14].

Theorem 3.3.13. *Fix $q \in (0,1)$ and $0 \leqslant \nu < 1$. Consider step initial data. Then for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, we have the "Mellin-Barnes-type" Fredholm determinant formula*

$$\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n};q\right)_\infty}\right] = \det\left(I+K_\zeta\right) \tag{3.36}$$

where $\det(I+K_\zeta)$ is the Fredholm determinant of $K_\zeta : L^2(C_1) \to L^2(C_1)$ for C_1 a positively oriented circle containing 1 with small enough radius so as to not contain 0, $1/q$ and $1/\nu$. The operator K_ζ is defined in terms of its integral kernel

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds$$

with

$$g(w) = \left(\frac{(\nu w;q)_\infty}{(w;q)_\infty}\right)^n \exp\left((q-1)t\sum_{k=0}^{\infty} \frac{R}{\nu}\frac{\nu w q^k}{1-\nu w q^k} - L\frac{wq^k}{1-wq^k}\right)\frac{1}{(\nu w;q)_\infty}.$$

The following "Cauchy-type" formula also holds:

$$\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n};q\right)_\infty}\right] = \frac{\det\left(I+\zeta\tilde{K}\right)}{(\zeta;q)_\infty}, \tag{3.37}$$

where $\det\left(I+\zeta\tilde{K}\right)$ is the Fredholm determinant of ζ times the operator $\tilde{K} : \mathbb{L}^2(C_{0,1}) \to \mathbb{L}^2(C_{0,1})$ for $C_{0,1}$ a positively oriented circle containing 0 and 1 but not $1/\nu$, and the operator $\tilde{K}$ is defined by its integral kernel

$$\tilde{K}(w,w') = \frac{g(w)/g(qw)}{qw'-w}.$$

Proof. We will sketch the main deductions which occur in the proof of the Mellin-Barnes type formula (3.36). Similar derivations (with all details given) of such Fredholm determinants from moment formulas can be found in [BC14, Theorem 3.18], [BCS14, Theorem 1.1] or more recently [Cor14, Theorem 1.10] and the proofs always follow the same general scheme (cf. [BCS14, Section 3.1]). Propositions 3.2 to 3.6 in [BCS14] show that for $|\zeta|$ small enough and C_1 a positively oriented circle containing 1 with small enough radius,

$$\sum_{k=0}^{\infty} \mathbb{E}\left[q^{k(x_n(t)+n)}\right]\frac{\zeta^k}{[k]_q!} = \det\left(I+K_\zeta\right), \tag{3.38}$$

with $[k]_q!$ as in (3.3). The only technical condition to verify is that

$$\sup\{|g(w)/g(wq^s)| :\ w \in C_1,\ k \in \mathbb{Z}_{>0},\ s \in D_{R,d,k}\} < \infty.$$

Here, $D_{R,d,k}$ is the contour depicted in [BCS14, Figure 3]. Note that here R is not the asymmetry parameter of the process but the radius of the circular part of the contour $D_{R,d,k}$. If one chooses R large enough, d small enough, and the radius of C_1 small enough, then $q^s w$ stay in a neighbourhood of the segment $[0, \sqrt{d}]$. The function g has singularities at q^{-n} and $\nu^{-1}q^{-n}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Hence for $w \in C_1$ a small but fixed circle around 1, one can choose R and d such that $q^s w$ stay in a compact region of the complex plane away from all singularities, and thus the ratio $|g(w)/g(wq^s)|$ remains bounded.

By an application of the q-binomial theorem (3.4), for $|\zeta| < 1$ we also have that

$$\sum_{k=0}^{\infty} \mathbb{E}\left[q^{k(x_n(t)+n)}\right] \frac{\zeta^k}{[k]_q!} = \mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right],$$

proving that (3.36) holds for $|\zeta|$ sufficiently small. Both sides of (3.36) can be seen to be analytic over $\mathbb{C} \setminus \mathbb{R}_+$. The left-hand side equals

$$\sum_{k=0}^{\infty} \frac{\mathbb{P}\big(x_n(t) + n = k\big)}{(1 - \zeta q^k)(1 - \zeta q^{k+1}) \cdots}.$$

For any $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ the infinite products are uniformly convergent and bounded away from zero on a neighbourhood of ζ, which implies that the series is analytic. The right-hand side of (3.36) is

$$\det(I + K_\zeta) = 1 = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{C_1} \mathrm{d}w_1 \ldots \int_{C_1} \mathrm{d}w_n \det\left(K_\zeta(w_i, w_j)\right)_{i,j=1}^{n}.$$

Due to exponential decay in $|s|$ in the integrand of K_ζ, $\det\left(K_\zeta(w_i, w_j)\right)_{i,j=1}^{n}$ is analytic in ζ for all $w_1, \ldots, w_n \in C_1$. Analyticity of the Fredholm expansion proceeds from absolute and uniform convergence of the series on a neighbourhood of $\zeta \notin \mathbb{R}_+$. This can be shown using that $|g(w)/g(wq^s)| < const$ for $w \in C_1$ and $s \in 1/2 + i\mathbb{R}$ and Hadamard's bound to control the determinant.

We do not prove explicitly the Cauchy-type Fredholm determinants but refer to the Section 3.2 in [BCS14] where a general scheme is explained to prove such formulas. □

3.3.2 Two-sided generalizations of q-TASEP

The limits of the q-Hahn weights when ν goes to zero and and when $\epsilon = (\mu - \nu)/(1 - q)$ goes to zero do not commute, thus several choices are possible in order to build two-sided version of the q-TASEP and the q-Boson process (see, e.g. [BCS14]).

Case $\nu = 0$ for q-Hahn AEP

The limit when we make first ϵ tend to zero corresponds to taking $\nu = 0$ in the rates $\phi^R_{q,\nu}$ and $\phi^L_{q,\nu}$. We have

$$\phi^R_{q,0}(j|m) = R\,(1 - q^m)\,\mathbb{1}_{\{j=1\}} \qquad \text{and} \qquad \phi^L_{q,0}(j|m) = \frac{L}{[j]_q} \frac{(q;q)_m}{(q;q)_{m-j}}. \tag{3.39}$$

Let us describe the associated exclusion process. Independently for each $n \geqslant 1$, the particle at location $x_n(t)$ jumps to $x_n(t)+1$ at rate $R(1-q^{\mathrm{gap}})$, the gap being here $x_{n-1}(t)-x_n(t)-1$, and jumps to the location $x_n - j$ at rate $L/[j]_q(q;q)_{\mathrm{gap}}/(q;q)_{\mathrm{gap}-j}$, for all $j \in \{1,\dots,x_n - x_{n+1}-1\}$, the gap being here $x_n(t)-x_{n+1}(t)-1$. All the result in Section 3.3.1 apply for the case $\nu = 0$, and one could study this system in more details by analyzing the Fredholm determinant formula of Theorem 3.3.13.

A motivation for studying this process is the observation that as q goes to 1,

$$\phi^R(j|m) \approx R(1-q)m\mathbb{1}_{\{j=1\}} \qquad \text{and} \qquad \phi^L(j|m) \approx L(1-q)m\mathbb{1}_{\{j=1\}}. \tag{3.40}$$

Thus, the rates on the left and on the right have the same expression at the first order in $1-q$, and the limit of this process when $q \to 1$ may be interesting. Several scaling limits are possible.

Another two-sided q-TASEP preserving the duality

When sending first ν to zero, we have already noted that

$$\varphi_{q,\mu,0}(j|m) = \mu^j(\mu;q)_{m-j}\begin{bmatrix} m \\ j \end{bmatrix}_q \quad \text{and} \quad \varphi_{q^{-1},\mu^{-1},\infty}(j|m) = \mathbb{1}_{j=m}.$$

Hence, one can set $\mu = (1-q)\epsilon$, $b = L\epsilon$, $a = 1 - L\epsilon$, $\tau = t\epsilon^{-1}$ and send ϵ to zero. This limit suggests the definition of continuous time Markov processes described as follows. In this two-sided q-TASEP, the particle at location $x_n(t)$ jumps to $x_n(t)+1$ at rate $1-q^{\mathrm{gap}}$, the gap being here $x_{n-1}(t)-x_n(t)-1$, and jumps to the location $x_{n+1}+1$ at rate L. The Boson system is defined in order to preserve the duality property: if $y_i(t)$ particles occupy site i then one particle jumps to site $i-1$ at rate $1-q^{y_i(t)}$ and y_i particles jump to site $i+1$ at rate L. Indeed, the exclusion process is dual to the Boson process with respect to the function H, but one one would need more involved boundary conditions to write the generator of the Boson system as a k-particle free evolution generator subject to two-body boundary conditions.

Remark 3.3.14. A third way to define a two-sided version of the q-TASEP consists in noticing that in the usual q-TASEP, jumps to the right have rate $(1-q)[\mathrm{gap}]_{\mathrm{q}}$, thus one could give rate proportional to $(q^{-1}-1)[\mathrm{gap}]_{\mathrm{q}^{-1}}$ to the jumps to the left. In this case as well, the duality between the exclusion and the zero-range processes is still preserved, but one needs additional boundary conditions to write the evolution of the zero-range system as a free evolution equation plus two-body boundary conditions. More precisely, one has [1] to impose $(\Delta_i^- - q\Delta_{i+1}^-)|_{n_i=n_{i+1}}u = 0$ for the jumps to the right, and $(\Delta_i^+ - q\Delta_{i+1}^+)|_{n_i=n_{i+1}}u = 0$ for the jumps to the left. These two conditions together impose that $q=1$ and u is symmetric. In the zero-range formulation, it corresponds to the non-interacting case where all particles perform independent random walks.

Remark 3.3.15. It is not always necessary for Bethe ansatz solvability that the true evolution equation can be factored as a free evolution equation subject to two-body boundary conditions. A case in which this does not happen is studied in [BCG14].

1. That is, if one is searching for boundary conditions that do not involve the asymmetry parameters. More sophisticated boundary conditions involving asymmetry parameters might work.

3.3.3 Degenerations to known systems

Totally asymmetric case

When $R = 1$ and $L = 0$, we say that we are in the totally asymmetric case. This degeneration of the q-Hahn AZRP was studied by Takeyama in [Tak14]. Indeed, the particle system defined in [Tak14] is a zero-range process defined on $\mathbb{Z}$ controlled by two parameters s and q. Particles move from site i to $i-1$ independently for each $i \in \mathbb{Z}$, and the rate at which j particles move to the left from a site occupied by m particles is given by

$$\frac{s^{j-1}}{[j]_q} \prod_{i=0}^{j-1} \frac{[m-i]_q}{1+s[m-1-i]_q}.$$

Setting $s = (1-q)\frac{\nu}{1-\nu}$, one notices that

$$\frac{s^{j-1}}{[j]_q} \prod_{i=0}^{j-1} \frac{[m-i]_q}{1+s[m-1-i]_q} \;=\; \phi^R_{q,\nu}(j|m).$$

Remark 3.3.16. The totally asymmetric q-Hahn AEP, is also the natural continuous time limit of the (discrete-time) q-Hahn TASEP, and it was already noticed in [Pov13] that letting $\mu \to \nu$ and rescaling time was the right way of defining such a continuous time limit.

Multiparticle asymmetric diffusion model

When $\nu = q$, the jump rates of the q-Hahn AZRP and AEP no longer depend on the gap between consecutive particles (or the number of particles on each site in the zero-range formulation). The rates are now given by $R/[j]_{q^{-1}}$ and $L/[j]_q$. The zero-range model with N particles is exactly the "multi-particle asymmetric diffusion model" introduced by Sasamoto and Wadati [2] in [SW98b] and further studied by Lee [Lee12] (see also [AKK99, AKK98]). For the corresponding exclusion process, we prove (by an asymptotic analysis of the Fredholm determinant in (3.36)) in Section 3.5 that the rescaled positions of particles converge to a Tracy-Widom law.

Push-ASEP

As we have already noticed in Section 3.3.2, it can be interesting to examine alternative description of exclusion processes by applying particle-hole inversion. Let us consider the q-Hahn AEP when $\nu = 0$ (see Section 3.3.2), and let further $q = 0$. The process obtained after particle-hole inversion is known. Indeed, when $\nu = q = 0$, $\phi^R(j|m) = \mathbb{1}_{j=1}$ and $\phi^L(j|m) = 1$ for all $m \geqslant 1$. This corresponds to the Push-ASEP introduced in [BF08], wherein convergence to the Airy process is proved.

3.4 Predictions from the KPZ scaling theory

In this section, we explain how asymptotics of our Fredholm determinant formula (Theorem 3.3.13) confirms the universality predictions from the physics literature KPZ

2. [SW98b] defined the model with the restriction that $R/L = q$.

scaling theory [KMHH92, Spo12]. Although the original paper [KMHH92] on the KPZ scaling theory deals only with so-called single step models and directed random polymers, the predictions can be straightforwardly adapted to any exclusion process. In particular, we compute the non-universal constants arising in one-point limit theorems for the two-sided q-Hahn process. In Section 3.5, we provide a rigorous confirmation in a particular case.

Following [Spo12], we present the predictions of KPZ scaling theory in the context of exclusion processes. Assume that the translation invariant stationary measures for an exclusion process on $\mathbb{Z}$ with local dynamics are precisely labelled by the density of particles ρ, where

$$\rho = \lim_{n\to\infty} \frac{1}{2n+1} \#\{\text{particles between } -n \text{ and } n\}.$$

We define the average steady-state current $j(\rho)$ as the expected number of particles going from site 0 to 1 per unit time, for a system distributed according to the stationary measure indexed by ρ. We also define the integrated covariance $A(\rho)$ as

$$A(\rho) = \sum_{j\in\mathbb{Z}} Cov(\eta_0, \eta_j),$$

where $\eta_0, \eta_j \in \{0,1\}$ are the occupation variables of the exclusion system at sites 0 and j, and the covariance is taken under the ρ-indexed stationary measure. One expects that the rescaled particle density $\varrho(x,\tau)$, given heuristically by

$$\varrho(x,\tau) = \lim_{\tau\to\infty} \mathbb{P}(\text{There is a particle at } \lfloor xt \rfloor \text{ at time } t\tau)$$

satisfies the conservation equation

$$\frac{\partial}{\partial \tau}\varrho(x,\tau) + \frac{\partial}{\partial x} j(\varrho(x,\tau)) = 0, \tag{3.41}$$

with initial condition which is $\varrho(x,0) = \mathbb{1}_{x<0}$ for the step initial condition. The solution of this PDE yields a law of large numbers for the position of particles. For $\kappa \geqslant 0$, if n and t go to infinity with $n = \lfloor \kappa t \rfloor$, then one has

$$\frac{x_n(t)}{t} \xrightarrow[t\to\infty]{a.s.} \pi(\kappa). \tag{3.42}$$

It turns out that instead of expressing π as a function of κ, it is more convenient to parametrize π and κ by the local density ρ. The existence of such a parametrization is given by the solution of the PDE (3.41): $\pi(\rho)$ is implicitly determined by $\rho = \varrho(\pi(\rho), 1)$ and $\kappa(\rho)$ is determined by $\pi(\kappa(\rho)) = \pi(\rho)$.

Let $\lambda(\rho) = -j''(\rho)$. For ρ such that $\lambda(\rho) \neq 0$, the KPZ class conjecture states that

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_n(t) - t\pi(\rho)}{\sigma(\rho)t^{1/3}} \geqslant x\right) \xrightarrow[t\to\infty]{} F_{GUE}(-x), \tag{3.43}$$

where $\pi(\rho) = \frac{\partial j(\rho)}{\partial \rho}$,

$$\sigma(\rho) := \left(\frac{\lambda(\rho)\big(A(\rho)\big)^2}{2\rho^3}\right)^{1/3}$$

and n goes to infinity with $n = \lfloor \kappa(\rho)t \rfloor$. The precise definition of F_{GUE} is given in Definition 3.5.1.

Remark 3.4.1. The magnitude of fluctuations in [Spo12, Equation (2.14)] slightly differs from our expression $\lambda(\rho)\left(A(\rho)\right)^2/(2\rho^3)$. This is because [Spo12] considers fluctuations of the height function. The fluctuation of the height function is twice the fluctuations of the integrated current. And the fluctuations of the current are, on average, ρ times the fluctuations of a tagged particle. Then, the quantities $j(\rho)$ and $A(\rho)$ defined in [Spo12] differs from ours by a factor 2 and 4 respectively. Moreover, since we consider step-initial condition with particles on the left, it is more convenient to drop the minus sign. That is why the scale $\Big(-\frac{1}{2}\lambda\big(A(\rho)\big)^2\Big)^{1/3}$ becomes $\Big(\lambda(\rho)\big(A(\rho)\big)^2/\big(2\rho^3\big)\Big)^{1/3}$.

3.4.1 Hydrodynamic limit

In the case of the continuous time q-Hahn AEP, there exist translation invariant and stationary measures μ_α indexed by a parameter $\alpha \in (0,1)$ such that the gaps between particles $(x_n - x_{n+1} - 1)$ are independent and identically distributed according to

$$\mu_\alpha(\text{gap} = m) = \alpha^m \frac{(\nu;q)_m}{(q;q)_m}\frac{(\alpha;q)_\infty}{(\alpha\nu;q)_\infty}. \tag{3.44}$$

These measures are obviously translation invariant. Let us explain why they are stationary: It is known [Pov13, Cor14] that this measures are stationary for the totally asymmetric q-Hahn TASEP. Thus they are also stationary for the totally asymmetric continuous time case. Since the family of measures μ_α is stable by inversion of the parameters q and ν, they are also stationary in the two-sided case.

From now on, we consider step initial data. Fix $q \in (0,1), \nu \in [0,1)$ and assume $L = 1 - R$, without loss of generality. Under the local stationarity hypothesis, the density ρ is given by

$$\rho = \frac{1}{1 + \mathbb{E}[\text{gap}]}. \tag{3.45}$$

Assuming that the gap is distributed according to (3.44), we find

$$\begin{aligned}
\mathbb{E}[\text{gap}] &= \sum_{m=0}^{\infty} m\alpha^m \frac{(\nu;q)_m}{(q;q)_m}\frac{(\alpha;q)_\infty}{(\alpha\nu;q)_\infty}, \\
&= \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\left(\frac{(\alpha\nu;q)_\infty}{(\alpha;q)_\infty}\right), \\
&= \frac{1}{\log(q)}\left(\Psi_q(\theta) - \Psi_q(\theta+V)\right);
\end{aligned}$$

where $\theta = \log_q(\alpha)$ and $V = \log_q(\nu)$. Hence, around a position where gaps between particles are distributed according to (3.44), the density ρ is related to the parameter θ through

$$\rho(\theta) = \frac{\log(q)}{\log(q) + \Psi_q(\theta) - \Psi_q(\theta+V)}. \tag{3.46}$$

Let us compute the average steady-state current $j(\rho)$. We have

$$\begin{aligned} j(\rho) &= \rho \cdot \mathbb{E}[\text{drift}] \\ &= \rho \cdot \sum_{m=0}^{\infty} \alpha^m \frac{(\nu;q)_m}{(q;q)_m} \frac{(\alpha;q)_\infty}{(\alpha\nu;q)_\infty} \left(\sum_{j=1}^{m} j\phi^R_{q,\nu}(j|m) - \sum_{j'=1}^{m} j'\phi^L_{q,\nu}(j'|m) \right), \\ &= \rho\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \left(R/\nu \; G_q(\alpha\nu) - LG_q(\alpha) \right), \\ &= \rho \frac{1-q}{\log(q)^2} \big(R/\nu \; \Psi'_q(\theta+V) - L\Psi'_q(\theta) \big); \end{aligned}$$

where we have used the q-binomial theorem (3.4) to sum over m in the second equality and we have used Lemma 3.2.1 for the third equality. The functions π, κ and σ that arise in limit theorems (3.42) and (3.43) are a priori functions of the density ρ, but given the formula (3.46), one can express all quantities as functions of the θ variable. Also, all quantities should depend on the parameters q, ν and R (we have assumed that $L = 1-R$). In the following, the quantities π, κ and σ are denoted $\pi_{q,\nu,R}(\theta)$, $\kappa_{q,\nu,R}(\theta)$ and $\sigma_{q,\nu,R}(\theta)$.

Computation of $\pi_{q,\nu,R}(\theta)$.

Let $\varrho : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ a solution of the conservation PDE (3.41), with initial data corresponding to step initial condition. If a law of large numbers holds as in (3.42), then we should have for all $t > 0$

$$\varrho(\pi_{q,\nu,R}(\theta)t, t) = \rho(\theta). \tag{3.47}$$

Taking derivatives with respect to θ in (3.47) yields

$$\pi'_{q,\nu,R}(\theta) \frac{\partial \varrho}{\partial x} \big(\pi_{q,\nu,R}(\theta)t, t\big) = \frac{\partial \rho(\theta)}{\partial \theta},$$

where $\pi'_{q,\nu,R}(\theta) := \frac{\mathrm{d}\pi_{q,\nu,R}(\theta)}{\mathrm{d}\theta}$. Taking derivative with respect to t in (3.47) yields

$$\pi_{q,\nu,R}(\theta) \frac{\partial \varrho}{\partial x} \big(\pi_{q,\nu,R}(\theta)t, t\big) + \frac{\partial \varrho}{\partial t} \big(\pi_{q,\nu,R}(\theta)t, t\big) = 0.$$

This implies

$$\pi'_{q,\nu,R}(\theta) \frac{\partial \varrho}{\partial t} \big(\pi_{q,\nu,R}(\theta)t, t\big) = -\pi_{q,\nu,R}(\theta) \frac{\partial \varrho(\theta)}{\partial \theta}. \tag{3.48}$$

On the other hand, we also expect

$$j\left(\varrho\big(\pi_{q,\nu,R}(\theta)t, t\big)\right) = j(\rho(\theta)). \tag{3.49}$$

Taking derivative with respect to θ in (3.49) yields

$$\pi'_{q,\nu,R}(\theta) \frac{\partial j(\varrho)}{\partial x} \big(\pi_{q,\nu,R}(\theta)t, t\big) = \frac{\partial j(\rho)(\theta)}{\partial \theta}, \tag{3.50}$$

where j is considered as a function of the variable ρ in the left-hand-side. Finally (3.48) and (3.50), together with the fact that ϱ solves the PDE (3.41), imply that

$$\pi_{q,\nu,R}(\theta) = \frac{\partial j(\rho(\theta))}{\partial \theta} \Big/ \frac{\partial \rho(\theta)}{\partial \theta},$$

which yields the formula

$$\pi_{q,\nu,R}(\theta) = \frac{1-q}{\log(q)^2}\left[R/\nu\left(\Psi_q'(\theta+V) + \Psi_q''(\theta+V)\frac{\log q + \Psi_q(\theta) - \Psi_q(\theta+V)}{\Psi_q'(\theta+V) - \Psi_q'(\theta)}\right)\right.$$
$$\left. -L\left(\Psi_q'(\theta) + \Psi_q''(\theta)\frac{\log q + \Psi_q(\theta) - \Psi_q(\theta+V)}{\Psi_q'(\theta+V) - \Psi_q'(\theta)}\right)\right]. \quad (3.51)$$

Computation of $\kappa_{q,\nu,R}(\theta)$

We are searching for a function $\kappa_{q,\nu,R}(\theta)$ such that asymptotically when t goes to infinity, the particle indexed by $\lfloor \kappa_{q,\nu,R}(\theta)t \rfloor$ is asymptotically at the position $\pi_{q,\nu,R}(\theta)t$, when the system starts from step initial condition. For this purpose we can integrate the density between the position $\pi(\theta)$ and a position where we know κ. We claim that since we start from step initial condition, at the left end of the rarefaction fan the density is continuous and equals 1, and thus $\alpha = 0$ (i.e. $\theta = +\infty$). We see later that a discontinuity on the right end of the rarefaction fan can occur. Since we consider step initial data, for any fixed t, $x_N(t) = -N$ for N large enough. Thus one has $\kappa(\theta = +\infty) = -\pi(\theta = +\infty)$. Integrating the density,

$$\kappa(\theta) - (-\pi(\infty)) = \int_\theta^\infty \rho(\theta')\ \frac{\mathrm{d}\pi(\theta')}{\mathrm{d}\theta'}\mathrm{d}\theta'.$$

where we have kept fixed the variables q, ν and R. Integrating by parts, this gives

$$\kappa(\theta) = -\pi(\infty) + [\rho\pi - j(\rho)]_\theta^{+\infty} = -\rho(\theta)\ \pi(\theta) + j(\rho)(\theta).$$

This yields the formula

$$\kappa_{q,\nu,R}(\theta) = \frac{1-q}{\log(q)}\frac{\frac{R}{\nu}\ \Psi_q''(\theta+V) - L\ \Psi_q''(\theta)}{\Psi_q'(\theta) - \Psi_q'(\theta+V)}. \quad (3.52)$$

In order to make sense physically, the quantity $\kappa_{q,\nu,R}(\theta)$ must be positive, at least for θ belonging to some interval $(\tilde{\theta}, +\infty)$. Since $\kappa_{q,\nu,R}(\theta)$ tends to $R-L$ when θ tends to infinity (equivalently $\alpha \to 0$), this requires that $R > L$ and suggests that the particles lie on a support of size $\mathcal{O}(\text{time})$ with high probability only if $R > L$.

Now assume that $R > L > 0$. Then $\kappa_{q,\nu,R}(\theta)$ tends to $-\infty$ when θ tends to 0. The local behaviour of particles around the first particles is described by the stationary measure μ_{α_0}, where $\alpha_0 = q^{\theta_0}$ is such that $\kappa_{q,\nu,R}(\theta_0) = 0$. If $R > L > 0$, then $0 < \theta_0 < \infty$, which means that the density of particles ρ is strictly positive around the first particle and that the density profile exhibit a discontinuity at the first particle, see Figure 3.3. (Note that the curved section in Figure 3.3 is the parametric curve $(\pi_{q,\nu}(\theta), \rho(\theta))$ for $\theta \in (\theta_0, +\infty)$ where θ_0 is such that $\kappa_{q,\nu}(\theta_0) = 0$. This density profile is proved as a consequence of Theorem 3.5.2 in the case $q = \nu$.) Figure 3.8 provides an additional confirmation using simulation data.

The macroscopic position of the first particle is then given by

$$\pi(\theta_0) = \frac{1-q}{(\log q)^2}\left(R/\nu\ \Psi_q'(\theta_0+V) - L\ \Psi_q'(\theta_0)\right),$$

where $\theta_0 = \log_q(\alpha_0)$. Not surprisingly, this is also the drift of a particle in an environment given by μ_{α_0}.

3.4.2 Magnitude of fluctuations

One first needs to compute $\lambda = -j''(\rho)$. We have expressions for $j(\rho(\theta))$ and $\rho(\theta)$ but we take the second derivative of the function j with respect to the variable ρ. We have that

$$
\begin{aligned}
j''(\rho(\theta)) = \frac{1-q}{(\log q)^3} \frac{(\log q + \Psi_q(\theta) - \Psi_q(\theta+V))^3}{(\Psi_q'(\theta) - \Psi_q'(\theta+V))^2} \times \\
\left(\frac{R}{\nu}\Psi_q'''(\theta+V) - L\Psi_q'''(\theta) - \left(\frac{R}{\nu}\Psi_q''(\theta+V) - L\Psi_q''(\theta) \right) \frac{\Psi_q''(\theta) - \Psi_q''(\theta+V)}{\Psi_q'(\theta) - \Psi_q'(\theta+V)} \right).
\end{aligned}
$$

Note that by Lemma 3.4.2, $j''(\rho) \neq 0$ so that the main assumption of the KPZ class conjecture is satisfied.

In order to compute $A(\rho)$, we follow [Spo12] and define

$$
Z(\alpha) = \frac{(\alpha\nu; q)_\infty}{(\alpha; q)_\infty}, \tag{3.53}
$$

the normalization constant in the definition of (3.44), and $G(\alpha) = \log(Z(\alpha))$. Then

$$
A = \frac{\alpha(\alpha G')'}{(1+\alpha G')^3},
$$

where all derivatives are taken with respect to the variable α. (The formula differs by a factor 4 with [Spo12] because we take occupation variables $\eta_i \in \{0,1\}$ instead of $\{-1,1\}$.) With Z as in (3.53), we have

$$
G'(\alpha) = \frac{1}{\alpha \log q} \left(\Psi_q(\theta) - \Psi_q(\theta+V) \right),
$$

and

$$
A(\theta) = \log q \frac{\Psi_q'(\theta) - \Psi_q'(\theta+V)}{(\log q + \Psi_q(\theta) - \Psi_q(\theta+V))^3}. \tag{3.54}
$$

Finally, $\sigma_{q,\nu}(\theta) = \left(\frac{\lambda A^2}{2\rho^3}\right)^{1/3}$ with

$$
\frac{\lambda A^2}{2\rho^3} = \frac{q-1}{4(\log q)^4} \left(\frac{R}{\nu}\Psi_q'''(\theta+V) - L\Psi_q'''(\theta) - \left(\frac{R}{\nu}\Psi_q''(\theta+V) - L\Psi_q''(\theta) \right) \frac{\Psi_q''(\theta) - \Psi_q''(\theta+V)}{\Psi_q'(\theta) - \Psi_q'(\theta+V)} \right). \tag{3.55}
$$

One should note that we have always $\sigma_{q,\nu}(\theta) > 0$ (see Lemma 3.4.2 for a proof of this claim).

3.4.3 Critical point Fredholm determinant asymptotics

We sketch an asymptotic analysis of the Mellin-Barnes Fredholm determinant formula of Theorem 3.3.13 that confirms the KPZ class conjecture for the continuous time two-sided q-Hahn exclusion process. In particular, we recover independently the functions $\pi_{q,\nu}(\theta)$, $\kappa_{q,\nu}(\theta)$ and $\sigma_{q,\nu}(\theta)$ from (3.51), (3.52) and (3.55). We do not provide all necessary justifications to make this rigorous. However, in Section 3.5, we do so for the $\nu = q$ case under certain ranges of parameters.

The function $x \mapsto 1/(-q^x;q)_\infty$ converges to 1 in $+\infty$ and 0 in $-\infty$ and the sequence of functions $\left(x \mapsto 1/(-q^{t^{1/3}x};q)_\infty\right)_{t>0}$ satisfies the hypotheses of Lemma 4.1.39 in [BC14]. On account of this, if we set

$$\zeta = -q^{-\kappa t - \pi t - t^{1/3}\sigma x},$$

then it follows that for $\sigma > 0$,

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n};q\right)_\infty}\right] = \lim_{t\to\infty} \mathbb{P}\left(\frac{x_n(t)-\pi t}{\sigma t^{1/3}} \geqslant x\right),$$

with $n = \lfloor \kappa t \rfloor$. For the moment, we let the constants κ, π and σ remain undetermined.

$\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n};q\right)_\infty}\right]$ is given by $\det(I + K_\zeta)$ as in (3.36). Assume for the moment that the contour C_1 for the variable w is a very small circle around 1. Let us make the change of variables

$$w = q^W, \quad w' = q^{W'}, \quad s + W = Z.$$

Then the Fredholm determinant can be written with the new variables as $\det(I + K_x)$ where K_x is an operator acting on $\mathbb{L}^2(C_0)$ where C_0 is the image of C_1 under the mapping $w \mapsto \log_q(w)$, defined by its kernel

$$K_x(W,W') = \frac{q^W \log q}{2\pi i} \int_{\mathcal{D}_W} \frac{\pi}{\sin(-\pi(Z-W))} \exp\Big(t\big(f_0(Z) - f_0(W)\big) - t^{1/3}\sigma x \log(q)(Z-W)\Big) \frac{1}{q^Z - q^{W'}} \frac{(\nu q^Z;q)_\infty}{(\nu q^W;q)_\infty} dZ, \tag{3.56}$$

where the new contour $\mathcal{D}_W$ as the straight line $W + 1/2 + i\mathbb{R}$, and the function f_0 is defined by

$$f_0(Z) = \kappa \log\left(\frac{(q^Z;q)_\infty}{(\nu q^Z;q)_\infty}\right) + \frac{1-q}{\log(q)}\left(\frac{R}{\nu}\Psi_q(Z+V) - L\left(\Psi_q(Z)\right)\right) - Z\log(q)\left(\kappa + \pi\right). \tag{3.57}$$

Since C_1 was any small enough circle around 1, C_0 can be deformed to be a small circle around 0, and we can also deform the contour for Z to be simply $1/2 + i\mathbb{R}$ without crossing any singularities.

The principle of Laplace's method is to deform the integration contours so that they go across a critical point of f_0, and then make a Taylor approximation around the critical point. Actually, we know that the Airy kernel would occur in the limit if this critical point is a double critical point, so we determine our unknown parameters (κ, π, σ) so as to have a double critical point. We have that

$$f_0'(Z) = \kappa\left(\Psi_q(Z+V) - \Psi_q(Z)\right) + \frac{1-q}{\log(q)}\left(\frac{R}{\nu}\Psi_q'(Z+V) - L\left(\Psi_q'(Z)\right)\right) - \log(q)\left(\kappa + \pi\right), \tag{3.58}$$

and

$$f_0''(Z) = \kappa\left(\Psi_q'(Z+V) - \Psi_q'(Z)\right) + \frac{1-q}{\log(q)}\left(\frac{R}{\nu}\Psi_q''(Z+V) - L\left(\Psi_q''(Z)\right)\right). \tag{3.59}$$

We see that if $\pi = \pi_{q,\nu}(\theta)$ and $\kappa = \kappa_{q,\nu}(\theta)$ as in (3.51) and (3.52), then $f_0'(\theta) = f_0''(\theta) = 0$. Hence, up to higher order terms in $(Z-\theta)$,

$$f_0(Z) - f_0(W) \approx \frac{f_0'''(\theta)}{6}\left((Z-\theta)^3 - (W-\theta)^3\right).$$

The next lemma, about the sign of f_0''', is proved in Section 3.5.3.

Lemma 3.4.2. *For any $q \in (0,1)$, $\nu \in [0,1)$, and any $R, L \geqslant 0$ such that $R + L = 1$, we have that for all $\theta > 0$, $f_0'''(\theta) > 0$.*

Using Lemma 3.4.2 we know the behaviour of $\mathfrak{Re}[f_0]$ in the neighbourhood of θ. To make Laplace's method rigorous, we must control the real part of f_0 along the contours for Z and W, and prove that only the integration in the neighbourhood of θ has a contribution to the limit. We do not prove that here, and the rest of the asymptotic analysis presented in this section would require some additional effort to be completely rigorous.

Assume that one is able to deform the contours for Z and W passing through θ so that

- The contour for Z departs θ with angles ξ and $-\xi$ where $\xi \in (\pi/6, \pi/2)$, and $\mathfrak{Re}[f_0]$ attains its maximum uniquely at θ,
- The contour for W departs θ with angles ω and $-\omega$ where $\omega \in (\pi/2, 5\pi/6)$, and $\mathfrak{Re}[f_0]$ attains its minimum uniquely at θ.

Then, modulo some estimates that we do not state explicitly here, the Fredholm determinant can be approximated by the following. We make the change of variables $Z-\theta = zt^{-1/3}$ and likewise for W and W'. Taking into account the Jacobian of the W and W' change of variables, we get that the kernel has rescaled to

$$\tilde{K}_x(w,w') = \frac{1}{2i\pi}\int \frac{1}{w-z}\frac{1}{z-w'}\exp\left(f_0'''(\theta)/6(z^3-w^3) - \sigma x\log(q)(z-w)\right)\mathrm{d}z. \tag{3.60}$$

Finally, if we set $\sigma = \left(\frac{-f_0'''(\theta)}{2(\log q)^3}\right)^{1/3}$, and we make the change of variables replacing $-z\sigma\log(q)$ by z and likewise for w and w', we get the kernel

$$\tilde{K}_x(w,w') = \frac{1}{2i\pi}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} \frac{1}{w-z}\frac{1}{z-w'}e^{z^3/3-w^3/3+x(z-w)}\mathrm{d}z, \tag{3.61}$$

acting on a contour coming from $\infty e^{-2i\pi/3}$ to $\infty e^{2i\pi/3}$ which does not intersect the contour for z. Let us call $\mathcal{G}$ this contour. Using the "$\det(I-AB) = \det(I-BA)$ trick" to reformulate Fredholm determinants, see e.g. Lemma 8.6 in [BCF14], one has that

$$\det(I+\tilde{K}_x)_{\mathbb{L}^2(\mathcal{G})} = \det(I-K_{\mathrm{Ai}})_{\mathbb{L}^2(-x,+\infty)},$$

where K_{Ai} is the Airy kernel defined in 3.5.1. Since $F_{\mathrm{GUE}}(x) = \det(I-K_{\mathrm{Ai}})_{\mathbb{L}^2(-x,+\infty)}$, we have that

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{x_n(t)-t\pi(\theta)}{\sigma(\theta)t^{1/3}} \geqslant x\right) \xrightarrow[t\to\infty]{} F_{GUE}(-x)$$

as claimed in (3.43).

The expression for $\sigma_{q,\nu}(\theta)$ in (3.55) is the same as $\sigma = \left(\frac{-f_0'''(\theta)}{2(\log q)^3}\right)^{1/3}$. Indeed, we have that

$$f_0'''(Z) = \frac{1-q}{\log q}\left(\frac{R}{\nu}\Psi_q'''(Z+V) - L\Psi_q'''(Z) - \left(\frac{R}{\nu}\Psi_q''(\theta+V) - L\Psi_q''(\theta)\right)\frac{\Psi_q''(Z)-\Psi_q''(Z+V)}{\Psi_q'(\theta)-\Psi_q'(\theta+V)}\right), \tag{3.62}$$

so that $(\sigma_{q,\nu}(\theta))^3 = \frac{-f_0'''(\theta)}{2(\log q)^3}$.

3.5 Asymptotic analysis

In this section, we make the arguments of Section 3.4.3 rigorous in the case $\nu = q$, which, in light of Section 3.3.3 corresponds with the MADM. In order to simplify the notations we set $\pi(\theta) = \pi_{q,q,R}(\theta)$, $\kappa(\theta) = \kappa_{q,q,R}(\theta)$, and $\sigma(\theta) = \sigma_{q,q,R}(\theta)$, without writing explicitly the dependency on the parameters q and R.

Definition 3.5.1. The distribution function $F_{\mathrm{GUE}}(x)$ of the GUE Tracy-Widom distribution is defined by $F_{\mathrm{GUE}}(x) = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)}$ where K_{Ai} is the Airy kernel,

$$K_{\mathrm{Ai}}(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{-2i\pi/3}\infty}^{e^{2i\pi/3}\infty} \mathrm{d}w \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \mathrm{d}z \frac{e^{z^3/3-zu}}{e^{w^3/3-wv}}\frac{1}{z-w},$$

where the contours for z comes from infinity with an angle $-\pi/3$ and go to infinity with an angle $\pi/3$; the contour for w comes from infinity with an angle $-2\pi/3$ and go to infinity with an angle $2\pi/3$, and both contours do not intersect.

Theorem 3.5.2. *Fix $q \in (0,1)$, $\nu = q$ and $R > L \geqslant 0$ with $R + L = 1$. Let $\theta > 0$ such that $\kappa(\theta) \geqslant 0$. Suppose additionally that $q^\theta > 2q/(1+q)$. Then, for $n = \lfloor \kappa(\theta)t \rfloor$, we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_n(t) - \pi(\theta)t}{\sigma(\theta)t^{1/3}} \geqslant x\right) = F_{\mathrm{GUE}}(-x).$$

Remark 3.5.3. One can check on simulated numerical data that the sign of $\sigma(\theta)$ must be positive. Indeed, on Figures 3.7 and 3.8, one can see that the simulated curve is above the limiting curve predicted from KPZ scaling theory. This is coherent with the positive sign of $\sigma(\theta)$ and the fact that the Tracy-Widom distribution has negative mean.

Theorem 3.5.4. *Fix $q \in (0,1)$, $\nu = q$ and let*

$$R_{min}(q) = \frac{q\Psi_q''\left(\log_q\left(\frac{2q}{1+q}\right)\right)}{\Psi_q''\left(\log_q\left(\frac{2q}{1+q}\right)\right) + q\Psi_q''\left(\log_q\left(\frac{2q^2}{1+q}\right)\right)} \in \left(\frac{1}{2}, 1\right).$$

Then for $R_{min}(q) < R < 1$ and $L = 1 - R$, there exists a real number $\theta_0 > 0$ such that $\kappa_{q,q,R}(\theta_0) = 0$, and we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_1(t) - \pi(\theta_0)t}{\sigma(\theta_0)t^{1/3}} \geqslant x\right) = F_{\mathrm{GUE}}(-x).$$

Remark 3.5.5. We expect the same kind of result for the fluctuations of the position of the first particle in any q-Hahn AEP with positive asymmetry, when the parameter ν is such that $0 < \nu < 1$.

Remark 3.5.6. The condition $q^\theta > 2q/(1+q)$ in Theorem 3.5.2 is most probably purely technical. It ensures that we do not cross any residues when deforming the integration contour in the definition of the kernel K_ζ in Theorem 3.3.13 (see Remark 3.5.8). The condition $R_{min}(q) < R$ in Theorem 3.5.4 is equivalent to $q^\theta > 2q/(1+q)$ in the particular setting of Theorem 3.5.4.

However, the condition $R < 1$ is really meaningful, since in the totally asymmetric case ($R = 1$), the first particle has Gaussian fluctuations.

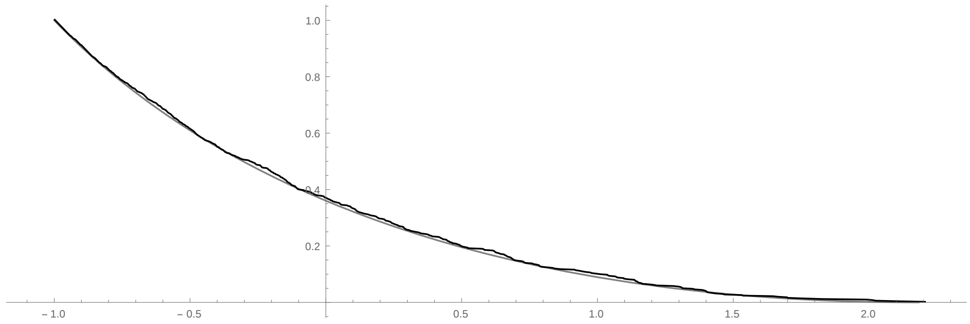

Figure 3.7: Comparison between simulated numerical data and predicted hydrodynamic limit. The black curve is $(x_N(t)/t, N/t)_N$ for N ranging from 1 to $t = 500$ (which is fixed) in the totally asymmetric case $(R = 1, L = 0)$, with $\nu = q = 0.4$. This is also the graph of the function $x \mapsto N_{tx}(t)/t$, where by definition $N_x(t)$ is the number of particles right to x at time t. The gray curve is the parametric curve $(\pi(\theta), \kappa(\theta))_{\theta\in(0,+\infty)}$ with $\pi(\theta)$ and $\kappa(\theta)$ as in (3.51) and (3.52).

3.5.1 Proof of Theorem 3.5.2

The proof uses Laplace's method and follows the style of [FV13] (similar proofs can be found in [Bar15] for q-TASEP with slow particles, in [BCF14] for the semi-discrete directed polymer, and in [Vet14] for the q-Hahn TASEP).

Fix $q \in (0, 1)$, $\nu = q$, $R > L \geqslant 0$ with $R + L = 1$ and $\theta > 0$ such that $\kappa(\theta) \geqslant 0$. In the particular case $q = \nu$, Theorem 3.3.13 states that for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n}; q\right)_\infty}\right] = \det\left(I + K_\zeta\right) \tag{3.63}$$

where $\det(I + K_\zeta)$ is the Fredholm determinant of $K_\zeta : L^2(C_1) \to L^2(C_1)$ for C_1 a positively oriented circle containing 1 with small enough radius so as to not contain 0, $1/q$. The operator K_ζ is defined in terms of its integral kernel

$$K_\zeta(w, w') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds \tag{3.64}$$

with

$$g(w) = \left(\frac{1}{1-w}\right)^n$$
$$\exp\left(\frac{(q-1)t}{\log(q)}\left(\frac{R}{q}\big(\Psi_q(W+1) + \log(1-q)\big) - L\big(\Psi_q(W) + \log(1-q)\big)\right)\right)\frac{1}{(qw; q)_\infty},$$

where $W = \log_q(w)$.

Remark 3.5.7. One notices that the argument of the exponential simplifies to $t\frac{(1-q)}{1+q}\frac{w}{1-w}$ when $R/L = q$. This yields a simpler analysis, though we work with the general R, L case here.

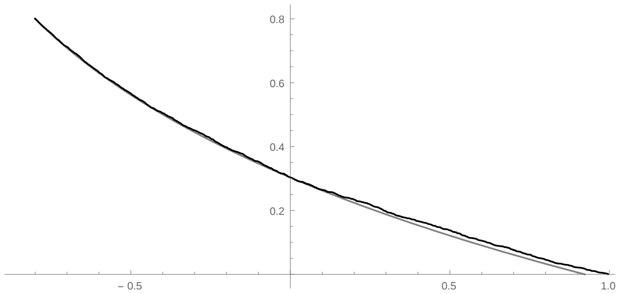

Figure 3.8: The black curve is a simulation of $(x_N(t)/t, N/t)_N$ for N ranging from 1 to $(R-L)t$, with $t = 1500$ fixed, $R = 0.9, L = .1$ and $\nu = q = 0.6$. The gray curve is the parametric curve $(\pi(\theta), \kappa(\theta))_{\theta\in(\theta_0,+\infty)}$ where θ_0 is such that $\kappa(\theta_0)$ as in Section 3.4.1. It goes from the point $(L-R, R-L)$ to the point $(\pi(\theta_0), 0)$. Since the slope of the curve $x \mapsto N_{tx}(t)/t$ (or equivalently $(x_N(t), N/t)_N$) is the macroscopic density $\rho(x,1)$, this simulationally confirms the discontinuity of density at the point $\pi(\theta_0)$ (see Figure 3.3).

As we have explained in Section 3.4.3, if we set

$$\zeta = -q^{-\kappa(\theta)t-\pi(\theta)t-t^{1/3}\sigma(\theta)x},$$

then it follows that

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{\left(\zeta q^{x_n(t)+n};q\right)_\infty}\right] = \lim_{t\to\infty} \mathbb{P}\left(\frac{x_n(t)-\pi(\theta)t}{\sigma(\theta)t^{1/3}} \geqslant x\right),$$

with $n = \lfloor \kappa(\theta) \rfloor$. Also from Section 3.4.3, making the change of variables:

$$w = q^W, \quad w' = q^{W'}, \quad s + W = Z,$$

The Fredholm determinant $\det(I + K_\zeta)$ equals $\det(I + K_x)$ where K_x is an operator acting on $\mathbb{L}^2(C_0)$ where C_0 is a small circle around 0, defined by its kernel

$$\begin{aligned} K_x(W,W') = \frac{q^W \log q}{2\pi i} \int_{\mathcal{D}} \frac{\pi}{\sin(-\pi(Z-W))} \\ \times \exp\Big(t\big(f_0(Z) - f_0(W)\big) - t^{1/3}\sigma(\theta)\log(q)x(Z-W)\Big) \frac{1}{q^Z - q^{W'}} \frac{(q^{Z+1};q)_\infty}{(q^{W+1};q)_\infty} dZ, \end{aligned} \tag{3.65}$$

where the new contour $\mathcal{D}$ is the straight line $1/2 + i\mathbb{R}$, and the function f_0 is defined by

$$f_0(Z) = \kappa(\theta)\log(1-q^Z) + \frac{1-q}{\log(q)}\left(\frac{R}{q}\Psi_q(Z+V) - L\Psi_q(Z)\right) - Z\log(q)\Big(\kappa(\theta)+\pi(\theta)\Big). \tag{3.66}$$

Using the expressions (3.52) and (3.51) for $\kappa(\theta)$ and $\pi(\theta)$ in terms of the q-digamma function, we have

$$\begin{aligned}f_0(Z) = \frac{1-q}{\log(q)}\Bigg(\frac{R}{q}\Big[\Psi_q(Z+1)+\log(1-q)-Z\Psi_q'(\theta+1)\\ +\frac{\Psi_q''(\theta+1)}{\log q}\left(\frac{(1-\alpha)^2}{\alpha}\frac{\log(1-q^Z)}{\log(q)}+Z(1-\alpha)\right)\Big]\\ -L\Big[\Psi_q(Z)+\log(1-q)-Z\Psi_q'(\theta)+\frac{\Psi_q''(\theta)}{\log q}\left(\frac{(1-\alpha)^2}{\alpha}\frac{\log(1-q^Z)}{\log(q)}+Z(1-\alpha)\right)\Big]\Bigg),\end{aligned}$$

with $\alpha = q^\theta$. For the derivatives, we have

$$\begin{aligned}f_0'(Z) = \frac{1-q}{\log(q)}\frac{R}{q}\left[\Psi_q'(Z+1)-\Psi_q'(\theta+1)+\frac{\Psi_q''(\theta+1)}{\log(q)}\left((1-\alpha)-\frac{(1-\alpha)^2}{\alpha}\frac{q^Z}{1-q^Z}\right)\right]\\ -\frac{1-q}{\log(q)}L\left[\Psi_q'(Z)-\Psi_q'(\theta)+\frac{\Psi_q''(\theta)}{\log(q)}\left((1-\alpha)-\frac{(1-\alpha)^2}{\alpha}\frac{q^Z}{1-q^Z}\right)\right],\end{aligned} \tag{3.67}$$

$$\begin{aligned}f_0''(Z) = \frac{1-q}{\log(q)}\frac{R}{q}\left[\Psi_q''(Z+1)-\frac{q^Z}{(1-q^Z)^2}\frac{(1-\alpha)^2}{\alpha}\Psi_q''(\theta+1)\right]\\ -\frac{1-q}{\log(q)}L\left[\Psi_q''(Z)-\frac{q^Z}{(1-q^Z)^2}\frac{(1-\alpha)^2}{\alpha}\Psi_q''(\theta)\right].\end{aligned}$$

Notice that the formulas become much simpler in the special case of Remark 3.5.7. Using the fact that $\Psi_q'(Z)-\Psi_q'(Z+1)=\log(q)^2\frac{q^Z}{1-q^Z}$, one has

$$f_0'(Z) = \frac{(1-q)\log(q)}{(1+q)(1-\alpha)^2}\left(\frac{q^Z}{1-q^Z}\left(1-\alpha^2-\frac{(1-\alpha)^2}{1-q^Z}\right)-\alpha^2\right). \tag{3.68}$$

One readily verifies that $f_0'(\theta)=f_0''(\theta)=0$. Since the saddle-point is at θ, we need to deform the integration contours for the variables Z and W so that they pass through θ and control the real part of f_0 along these contours.

Let $\mathcal{C}_\alpha$ be the positively oriented contour enclosing 0 defined by its parametrization

$$W(u) := \log_q(1-(1-\alpha)e^{iu}) \tag{3.69}$$

for $u\in(-\pi,\pi)$. Hence $q^{W(u)}$ ranges in a circle of radius $(1-\alpha)$ centered at 1 (see Figure 3.9). In order to use $\mathcal{C}_\alpha$ as the contour for W in the definition of the Fredholm determinant $\det(I+K_x)$, one should not encounter any singularities of the kernel when deforming the contour. Hence $\mathcal{C}_\alpha$ should not enclose -1 (this is the equivalent with the fact that the contour C_1 in Theorem 3.3.13 must not enclose $1/q$.) For the rest of this section, we impose the condition

$$2-\alpha < 1/q, \tag{3.70}$$

so that our contour deformation is valid.

When deforming the contour for the variable W, one also have to deform the contour for the variable Z, since in the original definition of K_ζ in Equation (3.64), the only singularities of the integrand for the variable s are for $s\in\mathbb{Z}$. This means that the singularities at $W+1, W+2,\dots$ for the variable Z must be on the right of the contour for

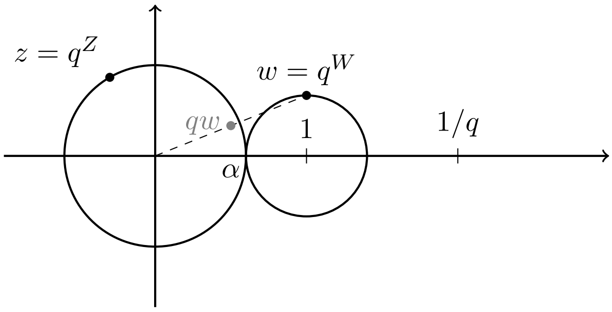

Figure 3.9: Images of the contours $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$ by the map $Z \mapsto q^Z$. The condition $\alpha > 2q/(1+q)$ is such that qw is always inside the image of $\mathcal{D}_\alpha$, which is the case in the figure.

Z. Let us choose the contour $\mathcal{D}_\alpha$ being the straight line parametrized by $Z(u) := \theta + iu$ for u in $\mathbb{R}$. To ensure that $\mathfrak{Re}[W+1] > \theta$, or equivalently that $qw < \alpha$ (see Figure 3.9), we impose the condition that

$$\alpha > \frac{2q}{1+q}. \tag{3.71}$$

Condition (3.71) implies in particular the previous condition $2 - \alpha < 1/q$.

Remark 3.5.8. Condition (3.71) is the same as condition (2.15) in [Vet14]. To get rid of this condition, one would need to add small circles around each pole in $W+1, W+2, \dots$ in the definition of the contour $\mathcal{D}$, as in [FV13]. The rest of the asymptotic analysis would remain almost unchanged provided one is able to prove that for any $W \in \mathcal{C}_\alpha$ and $k \geqslant 1$ such that $|q^{W+k}| > \alpha$, $\mathfrak{Re}[f_0(W) - f_0(W+k)] > 0$. In our case, it appears that the analysis of $\mathfrak{Re}[f_0(W) - f_0(W+k)]$ is computationally difficult and we do not pursue that here.

One notices that $\mathfrak{Re}[f_0]$ is periodic with a period $i\frac{2\pi}{\log q}$. Moreover, $f_0(\overline{Z}) = \overline{f_0(Z)}$ so that $\mathfrak{Re}[f_0]$ is determined by its restriction on the domain $\mathbb{R} + i[0, -\pi/\log q]$. The following results about the behaviour of $\mathfrak{Re}[f_0]$ along the contours are proved in Section 3.5.3.

Lemma 3.5.9. *For any $R > L \geqslant 0$ with $R + L = 1$, we have $f_0'''(\theta) > 0$.*

Proof. This is a particular case ($\nu = q$) of Lemma 3.4.2, which we prove in Section 3.5.2. □

Proposition 3.5.10. *Assume that (3.70) holds. For any $R > L \geqslant 0$ with $R + L = 1$, the contour $\mathcal{C}_\alpha$ is steep-descent for the function $-\mathfrak{Re}[f_0]$ in the following sense: the function $u \mapsto \mathfrak{Re}[f_0(W(u))]$ is increasing for $u \in [0, \pi]$ and decreasing for $u \in [-\pi, 0]$.*

Proposition 3.5.11. *Assume that (3.70) holds. For any $R > L \geqslant 0$ with $R + L = 1$, the contour $\mathcal{D}_\alpha$ is steep-descent for the function $\mathfrak{Re}[f_0]$ in the following sense: the function $t \mapsto \mathfrak{Re}[f_0(Z(u))]$ is decreasing for $u \in [0, -\pi/\log q]$ and increasing for $u \in [\pi/\log q, 0]$.*

We are now able to prove that asymptotically, the contribution to the Fredholm determinant of the contours are negligible outside a neighbourhood of θ.

Proposition 3.5.12. *For any fixed $\delta > 0$ and $\epsilon > 0$, there exists a real t_0 such that for all $t > t_0$*

$$\left|\det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_\alpha)} - \det(I + K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_{\alpha,\delta})}\right| < \epsilon$$

where $\mathcal{C}_{\alpha,\delta}$ is the intersection of $\mathcal{C}_\alpha$ with the ball $B(\theta, \delta)$ of radius δ around θ, and

$$K_{x,\delta}(W, W') = \frac{q^W \log q}{2\pi i} \int_{\mathcal{D}_\delta} \frac{\pi}{\sin(-\pi(Z - W))}$$
$$\times \exp\Big(t\big(f_0(Z) - f_0(W)\big) - t^{1/3}\sigma(\theta)\log(q)x(Z - W)\Big) \frac{1}{q^Z - q^{W'}} \frac{(q^{Z+1}; q)_\infty}{(q^{W+1}; q)_\infty} dZ,$$

where $\mathcal{D}_\delta = \mathcal{D} \cap B(\theta, \delta)$.

Proof. We have the Fredholm determinant expansion

$$\det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_\alpha)} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\pi}^{\pi} \mathrm{d}s_1 \dots \int_{-\pi}^{\pi} \mathrm{d}s_k \det\left(K_x(W(s_i), W(s_j))\right)_{i,j=1}^{k} \frac{\mathrm{d}W(s_i)}{\mathrm{d}s_i}, \tag{3.72}$$

with $W(s)$ as in (3.69). Let us denote by s_δ the positive real number such that $|W(s_\delta) - \theta| = \delta$. We need to prove that one can replace all the integrations on $[-\pi, \pi]$ by integrations on $[-s_\delta, s_\delta]$, making a negligible error. By Propositions 3.5.10 and 3.5.11, we can find a constant $c_\delta > 0$ such that for $|s| > s_\delta$ and for any $Z \in \mathcal{D}_\alpha$,

$$\mathfrak{Re}\big[f_0(Z) - f_0(W(s))\big] < -c_\delta.$$

The integral in (3.65) is absolutely integrable due to the exponential decay of the sine in the denominator. Thus, one can find a constant C_δ such that for $|s| > s_\delta$, any $W' \in \mathcal{C}_\alpha$ and t large enough,

$$\big|K(W(s), W')\big| < C_\delta \exp(-tc_\delta/2).$$

By dominated convergence the error (that is the expansion (3.72) with integration on $\big[-\pi, \pi\big]^k \setminus \big[-s_\delta, s_\delta\big]^k$) goes to zero for t going to infinity.

We also have to prove that one can localize the Z integrals as well. Recall that $\mathfrak{Re}[f_0]$ is periodic on the contour $\mathcal{D}_\alpha$. By the steep-descent property of Proposition 3.5.11 and the same kind of dominated convergence arguments, one can localize the integrations on

$$\bigcup_{k \in \mathbb{Z}} I_k, \text{ where } I_k = \big[\theta - i\delta + i2k\pi/\log q, \theta + i\delta + i2k\pi/\log q\big],$$

making a negligible error. Since $f_0(Z) - f_0(\theta) \approx f_0'''(\theta)/6(Z - \theta)^3$, by making the change of variables $Z = \theta + i2\pi k/\log q + zt^{-1/3}$, we see that only the integral for $Z \in [\theta - i\delta, \theta + i\delta]$ contributes to the limit. Indeed, for $k \neq 0$, and $Z \in I_k$

$$\frac{\mathrm{d}Z}{\sin(\pi(Z - W))} \approx t^{-1/3} \exp\left(-|2\pi^2 k/\log(q)|\right).$$

Hence the sum of contributions of integrals over I_k for $k \neq 0$ is $\mathcal{O}(t^{-1/3})$ and one can finally integrate over $\mathcal{D}_{W,\delta}$ making an error going to 0 as $t \to \infty$. It is not enough to show that the error made on the kernel goes to zero as t goes to infinity, but one can justify that the error on the Fredholm determinant goes to zero as well by a dominated convergence argument on the expansion (3.72). $\square$

By the Cauchy theorem, one can replace the contours $\mathcal{D}_\delta$ and $\mathcal{C}_{\alpha,\delta}$ by wedge-shaped contours $\hat{D}_{\varphi,\delta} := \{\theta + \delta e^{i\varphi sgn(y)}|y|; y \in [-1,1]\}$ and $\hat{C}_{\psi,\delta} := \{\theta + \delta e^{i(\pi-\psi)sgn(y)}|y|; y \in [-1,1]\}$, where the angles $\varphi, \psi \in (\pi/6, \pi/2)$ are chosen so that the endpoints of the contours do not change.

Let us make the change of variables

$$Z = \theta + \tilde{z}t^{-1/3},\ W = \theta + \tilde{w}t^{-1/3},\ W' = \theta + \tilde{w}'t^{-1/3}.$$

We define the corresponding rescaled contours

$$\mathcal{D}_\varphi^L := \{Le^{i\varphi sgn(y)}|y|; y \in [-1,1]\},$$
$$\mathcal{C}_\psi^L := \{Le^{i(\pi-\psi)sgn(y)}|y|; y \in [-1,1]\}.$$

Proposition 3.5.13. *We have the convergence*

$$\lim_{t\to\infty} \det(I + K_x)_{\mathbb{L}^2(\mathcal{C}_\alpha)} = \det(I + K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_\psi^\infty)},$$

where for $L \in \mathbb{R}_+ \cup \{\infty\}$,

$$K'_{x,L} = \frac{1}{2i\pi}\int_{\mathcal{D}_\varphi^L} \frac{d\tilde{z}}{(\tilde{z}-\tilde{w}')(\tilde{w}-\tilde{z})} \frac{\exp\left((-\tilde{z}\sigma(\theta)\log q)^3/3 - x\tilde{z}\sigma(\theta)\log q\right)}{\exp\left((-\tilde{w}\sigma(\theta)\log q)^3/3 - x\tilde{w}\sigma(\theta)\log q\right)}.$$

Proof. By the change of variables and the discussion about contours above,

$$\det(I + K_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_{\alpha,\delta})} = \det(I + K^t_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_\psi^{\delta t^{1/3}})}$$

where $K^t_{x,\delta}$ is the rescaled kernel

$$K^t_{x,\delta}(\tilde{w}, \tilde{w}') = t^{-1/3} K_{x,\delta}(\theta + \tilde{w}t^{-1/3}, \theta + \tilde{w}'t^{-1/3}),$$

where we use the contours $\mathcal{D}_\varphi^{\delta t^{1/3}}$ for the integration with respect to the variable $\tilde{z}$.

Let us estimate the error that we make by replacing f_0 by its Taylor approximation. We recall that with our definition of $\sigma(\theta)$ in (3.55),

$$f_0'''(\theta) = -2\left(\sigma(\theta)\log(q)\right)^3.$$

Using Taylor expansion, there exists C_{f_0} such that

$$|f_0(Z) - f_0(\theta) + \left(\sigma(\theta)\log(q)(Z-\theta)\right)^3/3| < C_{f_0}|Z-\theta|^4,$$

for Z in a fixed neighbourhood of θ (say e.g. $|Z-\theta| < \theta$). Hence for $Z = \theta + \tilde{z}t^{-1/3}$, $W = \theta + \tilde{w}t^{-1/3}$,

$$\begin{aligned}\Big|t\big(f_0(Z) - f_0(W)\big) - \big((-\sigma(\theta)\log(q)\tilde{z})^3/3 - (-\sigma(\theta)\log(q)\tilde{w})^3/3\big)\Big| <\\ t^{-1/3}C_{f_0}\left(|\tilde{z}|^4 + |\tilde{w}|^4\right) \leqslant \delta\left(|\tilde{z}|^3 + |\tilde{w}|^3\right). \quad (3.73)\end{aligned}$$

To control the other factors in the integrand, let

$$F(Z, W, W') := \frac{t^{-1/3}q^W \log(q)}{q^Z - q^{W'}} \frac{\pi t^{-1/3}}{\sin(\pi(Z-W))} \frac{(q^{Z+1};q)_\infty}{(q^{W+1};q)_\infty}.$$

we have that

$$F(Z, W, W') \xrightarrow[t\to\infty]{} F^{lim}(\tilde{z}, \tilde{w}, \tilde{w}') := \frac{1}{\tilde{z}-\tilde{w}'}\frac{1}{\tilde{z}-\tilde{w}}.$$

Lemma 3.5.14. *For $\tilde{z} \in \mathcal{D}_{\varphi}^{\delta t^{1/3}}$, and $\tilde{w}, \tilde{w}' \in \mathcal{C}_{\psi}^{\delta t^{1/3}}$, with $Z = \theta + \tilde{z}t^{-1/3}, W = \theta + \tilde{w}t^{-1/3}$ and $W' = \theta + \tilde{w}'t^{-1/3}$, we have that*

$$|F(Z, W, W') - F^{lim}(\tilde{z}, \tilde{w}, \tilde{w}')| < Ct^{-1/3}P(|\tilde{z}|, |\tilde{w}|, |\tilde{w}'|)F^{lim}(\tilde{z}, \tilde{w}, \tilde{w}'),$$

where and P is a polynomial and C is a constant independent of t and δ, as soon as δ belongs to some fixed neighbourhood of 0.

Proof. Since $|Z - \theta| < \delta, |W - \theta| < \delta$ and $|W' - \theta| < \delta$, there exist constants C_1, C_2 and C_3 such that

$$\left|\frac{q^W \log(q)(Z - W')}{q^Z - q^{W'}} - 1\right| \leqslant C_1(|Z - \theta| + |W' - \theta|),$$

$$\left|\frac{\pi(Z - W)}{\sin(\pi(Z - W))} - 1\right| \leqslant C_2(|Z - \theta| + |W - \theta|),$$

$$\left|\frac{(q^{Z+1}; q)_\infty}{(q^{W+1}; q)_\infty} - 1\right| \leqslant C_3(|Z - \theta| + |W - \theta|).$$

Hence there exists a constant C and a polynomial P of degree 3 such that

$$\left|\frac{F(Z, W, W')}{F^{lim}(\tilde{z}, \tilde{w}, \tilde{w}')} - 1\right| \leqslant Ct^{-1/3}P(|\tilde{z}|, |\tilde{w}|, |\tilde{w}'|),$$

and the result follows. $\square$

Now we estimate the difference between the kernels $K^t_{x,\delta}$ and $K'_{x,\delta t^{1/3}}$. Let

$$f(Z, W, W') = t\big(f_0(Z) - f_0(W)\big) - t^{1/3}\sigma(\theta)\log(q)x(Z - W)$$

and

$$f^{lim}(\tilde{z}, \tilde{w}, \tilde{w}') = \big((-\tilde{z}\sigma(\theta)\log q)^3/3 - x\tilde{z}\sigma(\theta)\log q\big) - \big((-\tilde{w}\sigma(\theta)\log q)^3/3 - x\tilde{w}\sigma(\theta)\log q\big).$$

The difference between the kernels is estimated by

$$\begin{aligned}\big|K^t_{x,\delta}(\tilde{w}, \tilde{w}') - K'_{x,\delta t^{1/3}}(\tilde{w}, \tilde{w}')\big| < &\int_{\mathcal{D}_{\varphi}^{\delta t^{1/3}}} \mathrm{d}\tilde{z} \exp(f^{lim})|F| \cdot |\exp(f^{lim} - f) - 1| \\ &+ \int_{\mathcal{D}_{\varphi}^{\delta t^{1/3}}} \mathrm{d}\tilde{z} \exp(f^{lim})|F - F^{lim}|, \quad (3.74)\end{aligned}$$

where we have omitted the arguments of the functions $f(Z, W, W')$, $f^{lim}(\tilde{z}, \tilde{w}, \tilde{w}')$, $F(Z, W, W')$ and $F^{lim}(\tilde{z}, \tilde{w}, \tilde{w}')$.

Using the inequality $|\exp(x) - 1| < |x|\exp(|x|)$ and (3.73), we have

$$\big|\exp(f^{lim} - f) - 1\big| < t^{-1/3}C_{f_0}\left(|\tilde{z}|^4 + |\tilde{w}|^4\right)\exp\left(\delta\left(|\tilde{z}|^3 + |\tilde{w}|^3\right)\right).$$

Hence, for δ small enough, the first integral in the right-hand-side of (3.74) have cubic exponential decay in $|\tilde{z}|$, and the limit when $t \to \infty$ is zero by dominated convergence. The second integral goes to zero as well by the same argument. We have shown pointwise convergence of the kernels. In order to show that the Fredholm determinants also converge, we give a dominated convergence argument. The estimate (3.73) also shows that for δ small enough, one can bound the kernel $K^t_{x,\delta}$ by

$$|K^t_{x,\delta}(\tilde{w},\tilde{w}')| < C\exp\left(\mathfrak{Re}[(\sigma(\theta)\log(q)\tilde{w}^3)]/6\right)$$

for some constant C. Then, Hadamard's bound yields

$$\det\left(K^t_{x,\delta}(\tilde{w}_i,\tilde{w}_j)\right)_{i,j=1}^n \leqslant n^{n/2}C^n\prod_{i=1}^n \exp\left(\mathfrak{Re}[\sigma(\theta)\log(q)\tilde{w}_i^3]/6\right).$$

It follows that the Fredholm determinant expansion

$$\det(I+K^t_{x,\delta})_{\mathbb{L}^2(\mathcal{C}_\psi^{\delta t^{1/3}})} = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathcal{C}_\psi^{\delta t^{1/3}}} \mathrm{d}\tilde{w}_1 \dots \int_{\mathcal{C}_\psi^{\delta t^{1/3}}} \mathrm{d}\tilde{w}_n \det\left(K^t_{x,\delta}(\tilde{w}_i,\tilde{w}_j)\right)_{i,j=1}^n,$$

is absolutely integrable and summable. Thus, by dominated convergence

$$\begin{aligned}\lim_{t\to\infty}\det(I+K_x)_{\mathbb{L}^2(\mathcal{C}_\alpha)} &= \lim_{t\to\infty}\det(I+K'_{x,\delta t^{1/3}})_{\mathbb{L}^2(\mathcal{C}_\psi^{\delta t^{1/3}})}\\ &= \det(I+K'_{x,\infty})_{\mathbb{L}^2(\mathcal{C}_\psi^{\infty})}.\end{aligned}$$

□

Finally, using a reformulation of the Airy kernel as in Section 3.4.3, and a new change of variables $\tilde{z} \leftarrow -z\sigma(\theta)\log q$, and likewise for $\tilde{w}$ and $\tilde{w}'$, one gets

$$\det(I+K'_{x,\infty}) = \det(I-K_{\mathrm{Ai}})_{\mathbb{L}^2(-x,+\infty)},$$

which finishes the proof of Theorem 3.5.2.

3.5.2 Proof of Theorem 3.5.4

The condition $R<1$ ensures that there exists a solution $\theta_0>0$ to the equation

$$\kappa_{q,q,R}(\theta)=0.$$

The condition $R>R_{min}(q)$ ensures that the solution θ_0 is such that $q^{\theta_0}>\frac{2q}{1+q}$. Indeed, given the definition of $\kappa_{q,\nu,R}(\theta)$ in (3.52), θ_0 satisfies

$$\frac{\Psi_q''(\theta_0+1)}{q\Psi_q''(\theta_0)} = \frac{1-R}{R}.$$

If we set $\theta_{max}=\log_q(2q/(1+q))$, then

$$\frac{\Psi_q''(\theta_{max}+1)}{q\Psi_q''(\theta_{max})} = \frac{1-R_{min}(q)}{R_{min}(q)}.$$

Since the function $\theta\mapsto\Psi_q''(\theta+1)/\Psi_q''(\theta)$ is increasing on $\mathbb{R}_+$, the condition $R>R_{min}(q)$ implies that $\theta_0<\theta_{max}$ and equivalently $q^{\theta_0}>\frac{2q}{1+q}$.

If we set $\zeta = -q^{-\pi(\theta_0)t - t^{1/3}\sigma(\theta_0)x}$, then

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{\left(\zeta q^{x_1(t)+1}; q\right)_\infty}\right] = \lim_{t\to\infty} \mathbb{P}\left(\frac{x_1(t) - \pi(\theta_0)t}{\sigma(\theta_0)t^{1/3}} \leqslant x\right).$$

The q-Laplace transform $\mathbb{E}\left[\frac{1}{\left(\zeta q^{x_1(t)+1}; q\right)_\infty}\right]$ is the Fredholm determinant of a kernel written in terms of f_0 exactly as in (3.65) with the only modification that the integrand should be multiplied by

$$\left(\frac{(\nu q^W; q)_\infty}{(q^W; q)_\infty}\right) \Big/ \left(\frac{(\nu q^Z; q)_\infty}{(q^Z; q)_\infty}\right).$$

This additional factor does not perturb the rest of the asymptotic analysis, and disappears in the limit when we rescale the variables around θ. Since the condition $q^{\theta_0} > 2q/(1+q)$ is satisfied, Theorem 3.5.4 follows from the proof of Theorem 3.5.2.

3.5.3 Proofs of Lemmas about properties of f_0

Proof of Lemma 3.4.2. With $R + L = 1$, the expression for $f_0'''(\theta)$ in Equation (3.62) is linear in R. Hence we may prove the positivity only for the extremal values, i.e. $R = 1$ and $R = 0$.

We first prove that the function

$$\theta \in \mathbb{R}_{>0} \mapsto \frac{\Psi_q'''(\theta)}{\Psi_q''(\theta)}$$

is strictly increasing. We show that the derivative is positive, that is for any $\theta > 0$,

$$\Psi_q''''(\theta)\Psi_q''(\theta) > \left(\Psi_q'''(\theta)\right)^2.$$

Using the series representation for the derivatives of the q-digamma function (3.8), this is equivalent to

$$\sum_{n,m\geqslant 1} \frac{n^4\alpha^n}{1-q^n}\frac{m^2\alpha^m}{1-q^m} > \sum_{n,m\geqslant 1} \frac{n^3\alpha^n}{1-q^n}\frac{m^3\alpha^m}{1-q^m}, \tag{3.75}$$

for $\alpha \in (0,1)$. Each side of (3.75) is a power series in α, and we claim that the inequality holds for each coefficient. Indeed, keeping only the coefficient of α^k, we have to prove that

$$\sum_{n=1}^{k-1} \frac{n^4(k-n)^2}{(1-q^n)(1-q^{k-n})} \geqslant \sum_{n=1}^{k-1} \frac{n^3(k-n)^3}{(1-q^n)(1-q^{k-n})}, \tag{3.76}$$

with strict inequality for at least one coefficient. Symmetrizing the left-hand-side, the inequality is equivalent to

$$\sum_{n=1}^{k-1} \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})}\frac{n^2+(k-n)^2}{2} \geqslant \sum_{n=1}^{k-1} \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})} n(k-n),$$

which clearly holds, with strict inequality for $k \geqslant 3$.

Case $R = 1$**.** In that case, we have to prove that

$$\Psi_q'''(\theta+V) - \Psi_q''(\theta+V)\frac{\Psi_q''(\theta) - \Psi_q''(\theta+V)}{\Psi_q'(\theta) - \Psi_q'(\theta+V)} < 0.$$

Using Cauchy mean value theorem, the ratio can be rewritten as

$$\frac{\Psi_q''(\theta)-\Psi_q''(\theta+V)}{\Psi_q'(\theta)-\Psi_q'(\theta+V)}=\frac{\Psi_q'''(\tilde\theta)}{\Psi_q''(\tilde\theta)},$$

for some $\tilde\theta\in(\theta,\theta+V)$. Since $\Psi_q''(x)<0$ for $x\in(0,+\infty)$, the inequality reduces to

$$\frac{\Psi_q'''(\theta+V)}{\Psi_q''(\theta+V)}>\frac{\Psi_q'''(\tilde\theta)}{\Psi_q''(\tilde\theta)},$$

which is true by the first part of the proof.

Case $R=0$. In that case, we have to prove that

$$\Psi_q'''(\theta)-\Psi_q''(\theta)\frac{\Psi_q''(\theta)-\Psi_q''(\theta+V)}{\Psi_q'(\theta)-\Psi_q'(\theta+V)}>0.$$

Using the same argument, one is left with proving

$$\frac{\Psi_q'''(\theta)}{\Psi_q''(\theta)}<\frac{\Psi_q'''(\tilde\theta)}{\Psi_q''(\tilde\theta)},$$

which is already done as well.

The proof also applies to the $\nu=0$ case, since the ν in the denominator in Equation (3.62) can be cancelled by a factor ν coming out from the q-digamma function. $\square$

Proof of Proposition 3.5.10. It suffices to prove that for $u\in(0,\pi)$,

$$\frac{\mathrm{d}}{\mathrm{du}}\mathfrak{Re}\big[f_0(W(u))\big]>0.$$

We have

$$\frac{\mathrm{d}}{\mathrm{du}}\mathfrak{Re}[f_0(W(u))]=\mathfrak{Re}\left[\frac{\mathrm{dW(u)}}{\mathrm{du}}f_0'(W(u))\right]=\mathfrak{Im}\left[\frac{1}{\log q}\frac{(1-\alpha)e^{iu}}{1-(1-\alpha)e^{iu}}f_0'(W(u))\right].$$

We use the linear dependence of f_0 on R as in the proof of Lemma 3.4.2.

Case $R=1$. Using (3.67), one needs to prove that

$$\mathfrak{Im}\left[\frac{\Psi_q'(W(u)+1)}{(\log q)^2}\frac{1-q^{W(u)}}{q^{W(u)}}-\frac{\Psi_q'(A+1)}{(\log q)^2}\frac{1-q^{W(u)}}{q^{W(u)}}+\frac{\Psi_q''(A+1)}{(\log q)^3}(1-\alpha)\frac{1-q^{W(u)}}{q^{W(u)}}\right]>0.$$

Using the series representation of the q-digamma function (3.5), the last inequality can be written as

$$\mathfrak{Im}\left[\sum_{k=1}^{\infty}\frac{(1-\alpha)e^{iu}}{1-(1-\alpha)e^{iu}}\left(\frac{(1-(1-\alpha)e^{iu})q^k}{(1-(1-(1-\alpha)e^{iu})q^k)^2}-\frac{\alpha q^k}{(1-\alpha q^k)^2}+\frac{\alpha q^k(1+\alpha q^k)(1-\alpha)}{(1-\alpha q^k)^3}\right)\right]>0$$

A computation – painful by hand, but easy for Mathematica – shows that the left-hand-side can be rewritten as

$$\sum_{k=1}^{\infty}\frac{4\sin(u)\sin^2(u/2)(1-\alpha)^2\alpha q^k(1-(2-\alpha)q^k)h(\alpha,q^k,u)}{|1-(1-\alpha)e^{iu}|^2|1-(1-(1-\alpha)e^{iu})q^k|^4(1-\alpha q^k)^3},\tag{3.77}$$

where

$$h(\alpha, q, u) = 1-\alpha q\Big(4-\alpha\big(2+2q(1-\alpha)+q^2(2-q)(1+(1-\alpha)^2)\big)\Big)+2(1-\alpha)\alpha^2 q^2(1-q)^2\cos(u).$$

For any $u \in (0,\pi)$, $\cos(u) \geqslant -1$, hence

$$h(\alpha, q, u) \geqslant 1-\alpha q(2-\alpha)\left(2-\alpha q^2(2-\alpha)(2-q)\right)$$

and for any $\alpha \in (0,1), q \in (0,1)$, $1-\alpha q(2-\alpha)\left(2-\alpha q^2(2-\alpha)(2-q)\right) \geqslant 0$. Thus, if $(2-\alpha)q < 1$, each term in (3.77) is positive.

Case $R = qL$. Since $R + L = 1$, this case corresponds to $R = q/(1+q)$ and $L = 1/(1+q)$. As we have noticed in Remark 3.5.7, we have the simpler expression (3.68) for f_0' when $R = qL$. Hence it is enough to show that

$$\mathfrak{Im}\left[\frac{1-q}{(1+q)(1-\alpha)^2}\left(1-\alpha^2-\frac{(1-\alpha)^2}{1-q^{W(u)}}-\alpha^2\frac{1-q^{W(u)}}{q^{W(u)}}\right)\right] > 0$$

or equivalently, that

$$\frac{1-q}{(1+q)(1-\alpha)^2}(1-\alpha)\sin(u)\left(1-\frac{\alpha^2}{|q^{W(u)}|^2}\right) > 0$$

which is true since $|q^{W(u)}| \leqslant \alpha$ by assumption.

To conclude, since f_0 is linear in R, the result is also proved for any value $R \in [q/(1+q), 1]$. □

Proof of Proposition 3.5.11. It suffices to show that for $u \in (0, -\pi/\log(q))$,

$$0 > \frac{\mathrm{d}}{\mathrm{d}u}\mathfrak{Re}[f_0(Z(u))] = \frac{-1}{\log q}\mathfrak{Im}[f_0'(Z(u))],$$

where $Z(u) = \theta + iu$ ($u \in \mathbb{R}$). We use the linear dependence of f_0 on R as in the proof of Lemma 3.4.2 and Proposition 3.5.10.

Case $R = 1$. Using (3.67), one has to show that

$$\mathfrak{Im}\left[\frac{\Psi_q'(Z(u)+1)}{(\log q)^2} - \frac{\Psi_q''(A+1)}{(\log q)^3}\frac{(1-\alpha)^2}{\alpha}\frac{q^{Z(u)}}{1-q^{Z(u)}}\right] > 0.$$

Using the series representation of the q-digamma function (3.5), the last inequality can be written

$$\mathfrak{Im}\left[\sum_{k=1}^{\infty}\frac{\alpha e^{iu}q^k}{(1-\alpha e^{iu}q^k)^2} - \frac{\alpha q^k(1+\alpha q^k)}{(1-\alpha q^k)^3}\frac{(1-\alpha)^2 e^{iu}}{1-\alpha e^{iu}}\right] > 0.$$

The left-hand-side equals

$$\sum_{k=1}^{\infty}\frac{\sin(u)\alpha(1-\alpha q^k)(2-\alpha-\alpha^2 q^k)(1+(\alpha-2)q^k)}{|1-\alpha e^{iu}q^k|^4\,(1-\alpha q^k)^3|1-\alpha e^{iu}|^2}. \tag{3.78}$$

If $(2-\alpha)q < 1$, then for all $k \geqslant 1$, $1+(\alpha-2)q^k \geqslant 0$, and each term in (3.78) is positive.

Case $R = qL$. Using (3.68), it is enough to show that

$$\mathfrak{Im}\left[\frac{q^{Z(u)}}{1-q^{Z(u)}}\left(1-\alpha^2-\frac{(1-\alpha)^2}{1-q^{Z(u)}}\right)-\alpha\right]>0,$$

which is true since the left-hand-side equals

$$\frac{2\sin(u)\alpha^2(1-\alpha^2)(1-\cos(u))}{|1-\alpha e^{iu}|^2}.$$

To conclude, since f_0 is linear in R, the result is also proved for any value $R \in [q/(1+q), 1]$. $\square$

Chapter 4

Beta Random walk in random environment

This chapter is based on the preprint [BC15b], written in collaboration with Ivan Corwin.

[BC15b] G. Barraquand and I. Corwin, *Random-walk in Beta-distributed random environment*, arXiv preprint arXiv:1503:04117 (2015).

Contents

We study an exactly solvable one-dimensional random walk in space-time i.i.d. random environment. It is a random walk on $\mathbb{Z}$ which performs nearest neighbour steps, according to transition probabilities following the Beta distribution and drawn independently at each time and each location. We call this model the Beta RWRE. Using methods of integrable probability, we find an exact Fredholm determinantal formula for the Laplace transform of the quenched probability distribution of the walker's position. An asymptotic analysis of this formula allows to prove a very precise limit theorem. It was already known that such a random walk satisfies a quenched large deviation principle [RASY13]. We show that for the Beta RWRE, the second order correction to the large deviation principle fluctuates on the cube-root scale with Tracy-Widom statistics. This brings the scope of KPZ universality to random walks in dynamic random environment, and the Beta RWRE is the first RWRE for which such a limit theorem has been proved. Moreover, our result translates in terms of the maximum of the locations of independent walkers in the same environment. Hence, the Beta RWRE can also be considered as a toy model for studying maxima of strongly correlated random variables.

Our route to discover the exact solvability of the Beta RWRE was through an equivalent directed polymer model with Beta weights, which is itself a limit of the q-Hahn TASEP (introduced in [Pov13] and further studied in [Cor14]). However, we show that the RWRE/polymer model can be analysed independently of its interacting particle system origin, via a rigorous variant of the replica method.

Our work generalizes a study of similar spirit, where a limit of the discrete-time geometric q-TASEP [BC13] was related to the strict weak lattice polymer [CSS15] (see also [OO15]). It should be emphasized that this procedure of translating the algebraic structure of interacting particle systems to directed polymer models was already fruitful in [BC14], where formulas for the q-TASEP allowed to study the law of continuous directed polymers related to the KPZ equation.

4.1 Definitions and main results

4.1.1 Random walk in space-time i.i.d. Beta environment

Definition 4.1.1. Let $(B_{x,t})_{x\in\mathbb{Z},t\in\mathbb{Z}_{\geqslant 0}}$ be a collection of independent random variables following the Beta distribution, with parameters α and β. We call this collection of random variables the environment of the walk. Recall that if a random variable B is drawn according to the $Beta(\alpha,\beta)$ distribution, then for $0\leqslant r\leqslant 1$,

$$\mathbb{P}\left(B\leqslant r\right)=\int_0^r x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\mathrm{d}x.$$

In this environment, we define the random walk in space-time Beta environment (abbreviated Beta-RWRE) as a random walk $(X_t)_{t\in\mathbb{Z}_{\geqslant 0}}$ in $\mathbb{Z}$, starting from 0 and such that

- $X_{t+1}=X_t+1$ with probability $B_{X_t,t}$ and
- $X_{t+1}=X_t-1$ with probability $1-B_{X_t,t}$.

A sample path is depicted in Figure 4.1. We denote by P and E (resp. $\mathbb{P}$ and $\mathbb{E}$) the measure and expectation associated to the random walk (resp. to the environment).

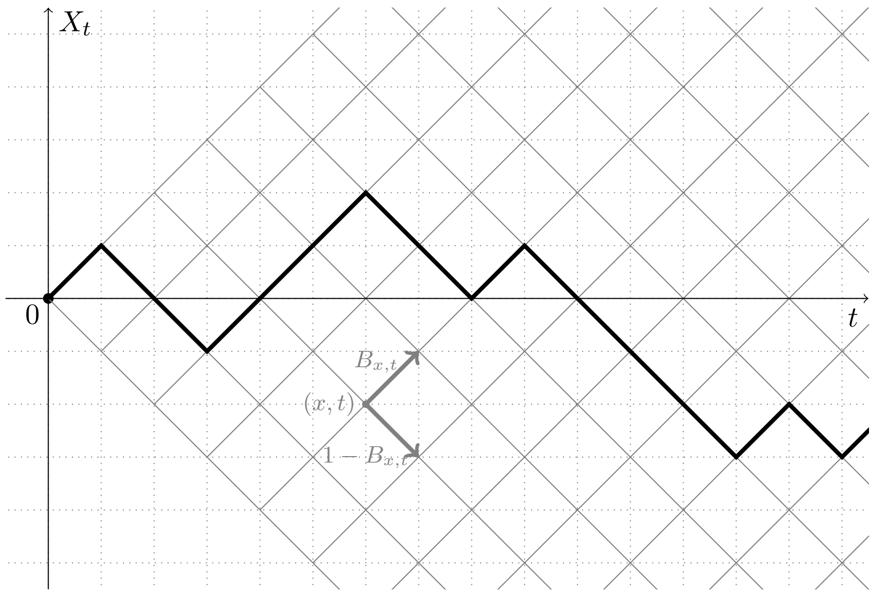

Figure 4.1: The graph of $t \mapsto X_t$ for the Beta RWRE. One sees that that the random walk $\mathbf{X}_t := (t, X_t)$ is also a (directed) random walk in a random environment in $\mathbb{Z}^2$.

Let $P(t,x) = \mathsf{P}(X_t \geqslant x)$. This is a random variable with respect to $\mathbb{P}$. Our first aim is to show that the Beta RWRE model is exactly solvable, in the sense that we are able to find the distribution of $P(t,x)$, by exploiting an exact formula for the Laplace transform of $P(t,x)$.

Remark 4.1.2. The random walk $(\mathbf{X}_t)_t$ in $\mathbb{Z}^2$, where $\mathbf{X}_t := (t, X_t)$ is a random walk in random environment in the classical sense, i.e. the environment is not dynamic (see Figure 4.1). It is a very particular case of random walk in Dirichlet random environment [ES06]. Dirichlet RWREs have generated some interest because it can be shown using connections between Dirichlet law and Pólya urn scheme that the annealed law of such random walks is the same as that of oriented-edge-reinforced random walks [ES02]. However, since the random walk $(\mathbf{X}_t)$ can go through a given edge of $\mathbb{Z}^2$ at most once, the connection to self-reinforced random walks is irrelevant for the Beta RWRE.

Remark 4.1.3.

- The Beta distribution with parameters $(1,1)$ is the uniform distribution on $(0,1)$.
- For B a random variable with $Beta(\alpha,\beta)$ distribution, $1-B$ is distributed according to a Beta distribution with parameters (β,α). Consequently, exchanging the parameters α and β of the Beta RWRE corresponds to applying a symmetry with respect to the horizontal axis.

4.1.2 Definition of the Beta polymer

Point to point Beta polymer

Definition 4.1.4. A point-to-point Beta polymer is a measure $Q_{t,n}$ on lattice paths π between $(0,0)$ and (t,n). At each site (s,k) the path is allowed to

- jump horizontally to the right from (s,k) to $(s+1,k)$,
- or jump diagonally to the upright from (s,k) to $(s+1,k+1)$.

An admissible path is shown in Figure 4.2. Let $B_{i,j}$ be independent random variables distributed according to the Beta distribution with parameters μ and $\nu-\mu$ where $0<\mu<\nu$. The measure $Q_{t,n}$ is defined by

$$Q_{t,n}(\pi)=\frac{\prod_{e\in\pi} w_e}{Z(t,n)}$$

where the weights w_e are defined by

$$w_e=\begin{cases} B_{ij} & \text{if } e=(i-1,j)\to(i,j)\\ 1 & \text{if } e=(i-1,i)\to(i,i+1)\\ 1-B_{i,j} & \text{if } e=(i-1,j-1)\to(i,j) \text{ with } i\geqslant j,\end{cases}$$

and $Z(t,n)$ is a normalisation constant called the partition function,

$$Z(t,n)=\sum_{\pi:(0,1)\to(t,n)}\prod w_e.$$

The free energy of the beta polymer is $\log Z(t,n)$. The partition function of the beta polymer satisfies the recurrence

$$\begin{cases} Z(t,n)=Z(t-1,n)B_{t,n}+Z(t-1,n-1)(1-B_{t,n}) & \text{for } t\geqslant n>1,\\ Z(t,t+1)=Z(t-1,t) & \text{for } t>0,\\ Z(t,1)=Z(t-1,1)B_{t,1} & \text{for } t>0.\end{cases} \tag{4.1}$$

with the initial data

$$Z(0,1)=1. \tag{4.2}$$

Remark 4.1.5. One recovers as $\nu\to\infty$ limit the strict-weak lattice polymer described in [OO15, CSS15]. As ν goes to infinity,

$$\nu\cdot \mathit{Beta}(\mu,\nu-\mu)\Rightarrow \mathit{Gamma}(\mu),$$

and $1-\mathit{Beta}(\mu,\nu-\mu)\Rightarrow 1$. There are $t-n+1$ horizontal edges in any admissible lattice path from $(0,1)$ to (t,n), and thus

$$\bar{Z}(t,n):=\lim_{\nu\to\infty}\nu^{t-n+1}Z(t,n)$$

is the partition function of the strict-weak polymer. Indeed, in the strict-weak polymer, the horizontal edges have weights $\mathit{Gamma}(\mu)$ whereas upright paths have weight 1.

Half-line to point Beta polymer

Another (equivalent) possible interpretation of the same quantity $Z(t,n)$ is the partition function of an ensemble of polymer paths starting from the "half-line" $\{(0,n):n>0\}$. Fix $t\geqslant 0$ and $n>0$. One considers paths starting from any point $(0,m)$ for $0<m\leqslant n$ and ending at (t,n). As for the point-to-point Beta polymer, paths are allowed to make

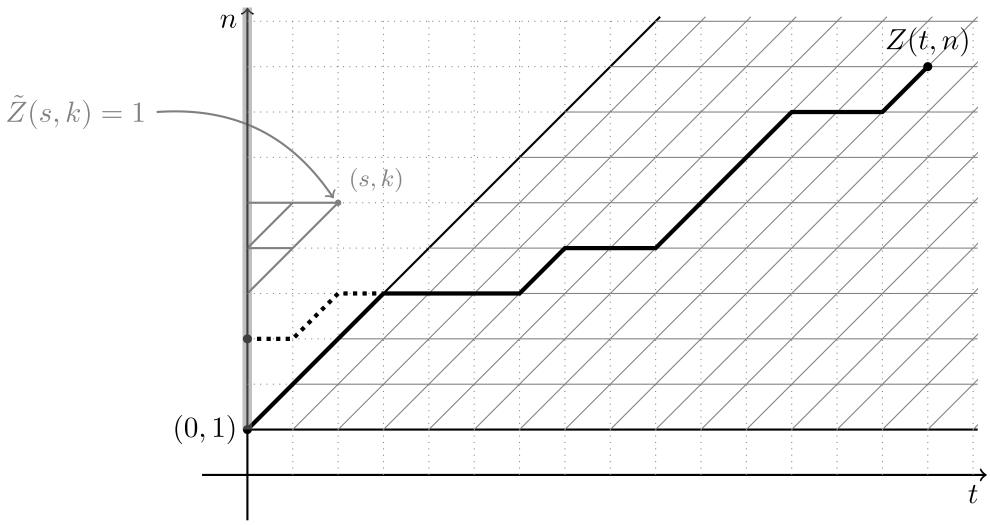

Figure 4.2: The thick line represents a possible polymer path in the point-to-point Beta polymer model. The dotted thick part represents a modification of the polymer path that is admissible if one considers the half-line to point polymer (see the paragraph 4.1.2). The partition function for the half-line to point model $\tilde{Z}(s,k)$ at the point (s,k) shown in gray equals 1.

right and diagonal steps. The weight of any path is the product of the weights of each edge along the path, and the weight $\tilde{w}_e$ of the edge e is now defined by

$$\tilde{w}_e = \begin{cases} B_{ij} & \text{if } e \text{ is the horizontal edge } (i-1,j)\to(i,j), \\ 1-B_{i,j} & \text{if } e \text{ is the diagonal edge } (i-1,j-1)\to(i,j) \text{ with } i\geqslant j. \end{cases}$$

Let us denote by $\tilde{Z}(t,n)$ the partition function in the half-line to point model. It is characterized by the recurrence

$$\tilde{Z}(t,n)=\tilde{Z}(t-1,n)B_{t,n}+\tilde{Z}(t-1,n-1)(1-B_{t,n})$$

for all $t,n>0$ and the initial condition $Z(0,n)=1$ for $n>0$. With the above definition of weights, we can see by recurrence that for any $t\geqslant 0$ and $n>t$, $\tilde{Z}(t,n)=1$. For example, in Figure 4.2, the possible paths leading to (s,k) are shown in gray. On the figure, one has

$$\tilde{Z}(s,k)=\tilde{Z}(2,6)=B_{1,6}B_{2,6}+(1-B_{1,6})B_{2,6}+B_{1,5}(1-B_{2,6})+(1-B_{1,5})(1-B_{2,6})=1.$$

Consequently, the partition functions of the half-line-to-point and the point-to-point model coincide for $t+1\geqslant n$. In the following, we drop the tilde above Z, even when considering the half-line-to point model, since the models are equivalent.

By deforming the lattice so that admissible path are up/right, and reverting the orientation of the path, one sees that the Beta polymer and the Beta-RWRE are closely related models, in the sense of Proposition 4.1.6. This proposition is proved in Section 4.2.3.

Proposition 4.1.6. *Consider the Beta-RWRE with parameters $\alpha,\beta>0$ and the Beta polymer with parameters $\mu=\alpha$ and $\nu=\alpha+\beta$. For any fixed $t,n\in\mathbb{Z}_{\geqslant 0}$ such that*

$t+1 \geqslant n$, then we have the equality in law

$$Z(t,n) = P\big(t, t-2n+2\big).$$

Moreover, conditioning on the environment of the Beta polymer corresponds to conditioning on the environment of the Beta RWRE.

4.1.3 Bernoulli-Exponential directed first passage percolation

Let us introduce the "zero-temperature" counterpart of the Beta RWRE.

Definition 4.1.7. Let (E_e) be a family of independent exponential random variables indexed by the horizontal and vertical edges e in the lattice $\mathbb{Z}^2$, such that E_e is distributed according to the exponential law of parameter a if e is a vertical edge and E_e is distributed according to the Exponential law of parameter b if e is a horizontal edge. Let $(\xi_{i,j})$ be a family of independent Bernoulli random variables with parameter $b/(a+b)$. For an edge e of the lattice $\mathbb{Z}^2$, we define the passage time t_e by

$$t_e = \begin{cases} \xi_{i,j} E_e \text{ if } e \text{ is the vertical edge } (i,j) \to (i,j+1), \\ (1-\xi_{i,j}) E_e \text{ if } e \text{ is the horizontal edge } (i,j) \to (i+1,j). \end{cases} \tag{4.3}$$

The first passage-time $T(n,m)$ in the Bernoulli-Exponential first passage percolation model is given by

$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e,$$

where the minimum is taken over all up/right path π from $(0,0$ to $D_{n,m}$, which is the set of points

$$D_{n,m} = \Big\{(i, n+m-i) : 0 \leqslant i \leqslant n\Big\}.$$

The percolation cluster $C(t)$ is defined by

$$C(t) = \Big\{(n,m) : T(n,m) \leqslant t\Big\}.$$

It can be constructed in a dynamic way (see Figure 4.4). At each time t, $C(t)$ is the union of points visited by (portions of) several directed up/right random walks in the quarter plane $\mathbb{Z}^2_{\geqslant 0}$. The evolution is as follows:

- At time 0, the percolation cluster contains the points of the path of a directed random walk starting from $(0,0)$.
  Indeed, since for any i,j, $\xi_{i,j}$ is a Bernoulli random variable in $\{0,1\}$, either the passage time from (i,j) to $(i+1,j)$ is zero, or the passage time from (i,j) to $(i,j+1)$ is zero. This implies that there exist a unique infinite up-right path starting from $(0,0)$ with zero passage-time. This path is distributed as a directed random walk.
- At time t, from each point in the percolation cluster where a random walk can branch, we add to the percolation cluster after an exponentially distributed waiting time, the path of that random walk. Paths starting with a vertical (resp. horizontal) edge are added at rate a (resp. b). This random walk almost surely crosses the percolation cluster somewhere, and we add to the percolation cluster only the points of the walk path up to the first hitting point.

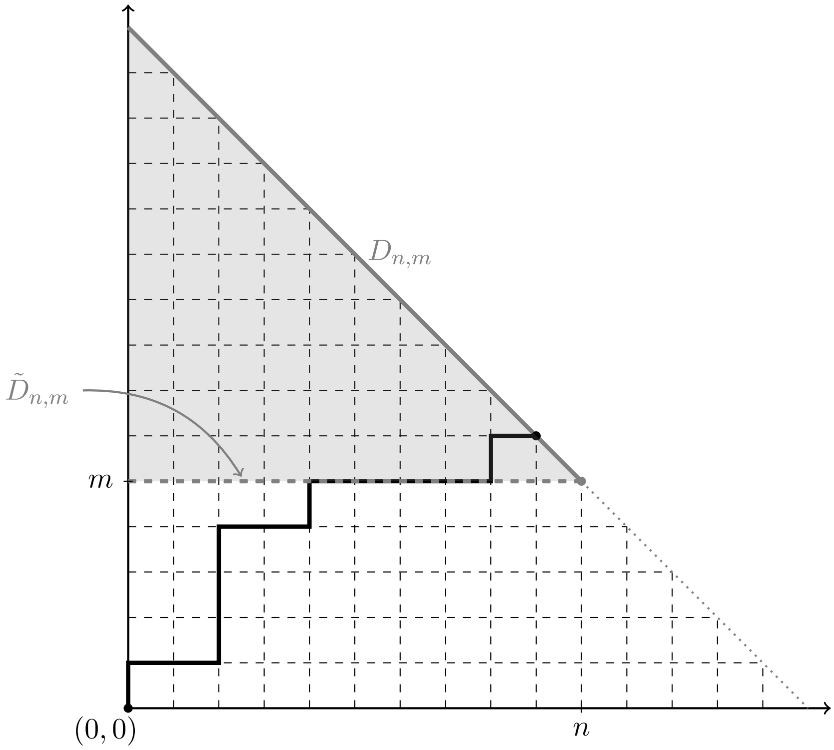

Figure 4.3: An admissible path for the Bernoulli-Exponential FPP model is shown on the figure. $T(n, m)$ is the passage time between $(0, 0)$ and $D_{n,m}$ (thick gray line). Note that the first passage time to $D_{n,m}$ is also the first passage time to $\tilde{D}_{n,m}$ depicted in dotted gray on the figure (cf Remark 4.1.8).

Indeed, any edge $e = (x, y)$ from a point x inside $C(t)$ to a point y outside $C(t)$, has a positive passage time. Hence, one adds the point y to the percolation cluster after an exponentially distributed waiting time t_e. Once the point y is added, one immediately adds to $C(t)$ all the points that one can reach from y with zero passage time. These points form a portion of random walk that will almost surely coalesce with the initial random walk path $C(0)$.

Remark 4.1.8. Denote by $\tilde{D}_{n,m}$ the set of points $\big\{(i, m) \ : 0 \leqslant i \leqslant n\big\}$ (see Figure 4.3). Any path going from $(0, 0)$ to $D_{n,m}$ has to go through a point of $\tilde{D}_{n,m}$. Moreover, the first passage time from any point of $\tilde{D}_{n,m}$ to the set $D_{n,m}$ is zero. Hence the first passage time from $(0, 0)$ to $\tilde{D}_{n,m}$ is also $T(n, m)$.

Remark 4.1.9. When b tends to infinity, E_e tends to 0 for all vertical edges, and one recovers the first passage percolation model introduced in [OCo99], which is the zero temperature limit of the strict-weak lattice polymer as explained in [OO15, CSS15].

Let us show how the Bernoulli-Exponential first passage percolation model is a limit of the Beta RWRE.

Proposition 4.1.10. *Let $\alpha_\epsilon = \epsilon a$ and $\beta_\epsilon = \epsilon b$. Let $P_\epsilon(t, x)$ be the probability distribution function of the Beta-RWRE with parameters α_ϵ and β_ϵ and $T(n, m)$ the first-passage time in the Bernoulli-Exponential FPP model with parameters a, b. Then, for all $n, m \geqslant 0$, $-\epsilon \log(P_\epsilon(n + m, m - n))$ weakly converges as ϵ goes to zero to $T(n, m)$, the first passage time from $(0, 0)$ to $D_{n,m}$ in the Bernoulli-Exponential FPP model.*

Proposition 4.1.10 is proved in Section 4.4.

4.1.4 Exact formulas

Our first result is an exact formula for the mixed moments of the polymer partition function $\mathbb{E}\big[Z(t, n_1) \cdots Z(t, n_k)\big]$. In light of Proposition 4.2.1, this result can be seen as a limit when q goes to 1 of the formula from Theorem 1.8 in [Cor14]. Even so, we prove this in an independent way in Section 4.3 via a rigorous polymer replica trick methods (See Proposition 4.3.4).

Proposition 4.1.11. *For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$, one has the following moment formula,*

$$\mathbb{E}\Big[Z(t, n_1) \cdots Z(t, n_k)\Big] = \\ \frac{1}{(2i\pi)^k} \int \cdots \int \prod_{1 \leqslant A < B \leqslant k} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} \left(\frac{\nu + z_j}{z_j}\right)^{n_j} \left(\frac{\mu + z_j}{\nu + z_j}\right)^{t} \frac{\mathrm{d}z_j}{\nu + z_j}, \quad (4.4)$$

where the contour for z_k is a small circle around the origin, and the contour for z_j contains the contour for $z_{j+1} + 1$ for all $j = 1, \ldots, k - 1$, as well as the origin, but all contours exclude $-\nu$.

The previous proposition provides a formula for the moments of the partition function $Z(t, n)$. Using tools developed in the study of Macdonald processes [BC14] (See also [Dot10, CDR10]), one is able to take the moment generating series, which yields a Fredholm determinant representation for the Laplace transform of $Z(t, n)$. We refer to [BC14, Section 3.2.2] for background about Fredholm determinants.

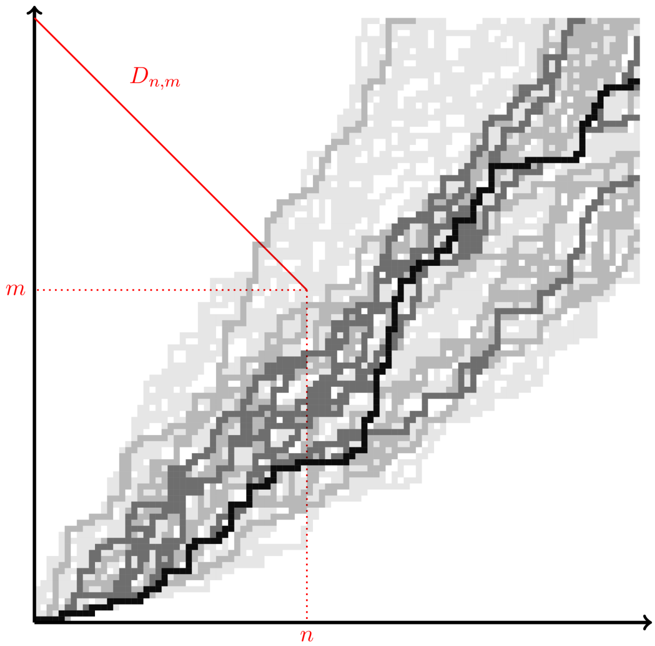

Figure 4.4: Percolation cluster for the Bernoulli-Exponential model with parameters $a = b = 1$ in a grid of size 100×100. The different shades of gray correspond to different times: the black line corresponds to the percolation cluster at time 0 and the other shades of gray corresponds to times 0.2, 0.5 and 1.2. This implies that for n and m chosen as on the figure, $0.2 \leqslant T(n, m) \leqslant 0.5$.

Theorem 4.1.12. *For $u \in \mathbb{C} \setminus \mathbb{R}_+$, fix $n, t \geqslant 0$ with $n \leqslant t+1$ and $\nu > \mu > 0$. Then one has*

$$\mathbb{E}\left[e^{uZ(t,n)}\right] = \det(I + K_u^{\mathrm{BP}})_{\mathbb{L}^2(C_0)}$$

where C_0 is a small positively oriented circle containing 0 but not $-\nu$ nor -1, and $K_u^{\mathrm{BP}} : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

$$K_u^{\mathrm{BP}}(v, v') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+s)} \frac{\mathrm{d}s}{s+v-v'}$$

where

$$g^{\mathrm{BP}}(v) = \left(\frac{\Gamma(v)}{\Gamma(\nu+v)}\right)^n \left(\frac{\Gamma(\nu+v)}{\Gamma(\mu+v)}\right)^t \Gamma(\nu+v). \tag{4.5}$$

In light of the relation between the Beta RWRE and the Beta polymer given in Proposition 4.1.6, we have a similar Fredholm determinant representation for the Laplace transform of $P(t,x)$.

Theorem 4.1.13. *For $u \in \mathbb{C} \setminus \mathbb{R}_+$, fix $t \in \mathbb{Z}_{\geqslant 0}$, $x \in \{-t, \dots, t\}$ with the same parity, and $\alpha, \beta > 0$. Then one has*

$$\mathbb{E}\left[e^{uP(t,x)}\right] = \det(I + K_u^{\mathrm{RW}})_{\mathbb{L}^2(C_0)} \tag{4.6}$$

where C_0 is a small positively oriented circle containing 0 but not $-\alpha-\beta$ nor -1, and $K_u^{\mathrm{RW}} : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

$$K_u^{\mathrm{RW}}(v, v') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g^{\mathrm{RW}}(v)}{g^{\mathrm{RW}}(v+s)} \frac{\mathrm{d}s}{s+v-v'}$$

where

$$g^{\mathrm{RW}}(v) = \left(\frac{\Gamma(v)}{\Gamma(\alpha+v)}\right)^{(t-x)/2} \left(\frac{\Gamma(\alpha+\beta+v)}{\Gamma(\alpha+v)}\right)^{(t+x)/2} \Gamma(v).$$

4.1.5 Limit theorem for the random walk

A quenched large deviation principle is proved in [RASY13, Section 4] for a wide class of random walks in random environment that includes the Beta-RWRE model. More precisely, the setting of [RASY13] applies to the random walk $\mathbf{X}_t = (t, X_t)$ (see Remark 4.1.2). The condition that one has to check is that the logarithm of the probability of each possible step has nice properties with respect to the environment (The random variables must belong to the class $\mathcal{L}$ defined in [RASY13, Definition 2.1]). Using the fact that if B is a $Beta(\alpha, \beta)$ random variable, $\log(B)$ and $\log(1-B)$ have integer moments of any order, [RASY13, Lemma A.4] ensures that the condition is satisfied. The limit

$$\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\left(\mathsf{E}\left[e^{zX_t}\right]\right)$$

exists $\mathbb{P}$-almost surely. Let I be the Legendre transform of λ. Then, we have [RASY13, Section 4] that for $x > (\alpha-\beta)/(\alpha+\beta)$,

$$\lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{P}(X_t > xt)\Big) = -I(x) \quad \mathbb{P} \text{ a.s.} \tag{4.7}$$

Remark 4.1.14. In the language of polymers, the limit (4.7) states the existence of the quenched free energy. Theorem 4.3 in [RAS14] states that for such random walks in random environment, we have that

$$\lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{P}(X_t = \lfloor xt \rfloor)\Big) = \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{P}(X_t > xt)\Big) = -I(x).$$

In other terms, the point-to-point free energy and the point-to-half-line free energies are equal.

In [RASY13, Theorem 3.1], a formula is given for I in terms of a variational problem over a space of measures. We provide a closed formula in the present case. It would be interesting to see how the variational problem recovers the formulas that we now present.

For the Beta-RWRE, critical point Fredholm determinant asymptotics shows that the function I is implicitly defined by

$$x(\theta) = \frac{\Psi_1(\theta+\alpha+\beta) + \Psi_1(\theta) - 2\Psi_1(\theta+\alpha)}{\Psi_1(\theta) - \Psi_1(\theta+\alpha+\beta)} \tag{4.8}$$

and

$$I(x(\theta)) = \frac{\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta+\alpha)}{\Psi_1(\theta) - \Psi_1(\theta+\alpha+\beta)}\Big(\Psi(\theta+\alpha+\beta)-\Psi(\theta)\Big)+\Psi(\theta+\alpha+\beta)-\Psi(\theta+\alpha), \tag{4.9}$$

where Ψ is the digamma function ($\Psi(z) = \Gamma'(z)/\Gamma(z)$) and Ψ_1 is the trigamma function ($\Psi_1(z) = \Psi'(z)$). The parameter θ does not seem natural at a first sight. It is convenient to use it as it will turn out to be the position of the critical point in the asymptotic analysis. When θ ranges from 0 to $+\infty$, $x(\theta)$ ranges from 1 to $(\alpha-\beta)/(\alpha+\beta)$. This covers all the interesting range of large deviation events since $(\alpha-\beta)/(\alpha+\beta)$ is the expected drift of the random walk, and we know that $\mathsf{P}(X_t > xt) = 0$ for $x > 1$.

Moreover, we define $\sigma(\theta) > 0$ such that

$$2\sigma(\theta)^3 = \Psi_2(\theta+\alpha)-\Psi_2(\alpha+\beta+\theta)+\frac{\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)}{\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)}\left(\Psi_2(\alpha+\beta+\theta) - \Psi_2(\theta)\right). \tag{4.10}$$

In the case $\alpha = \beta = 1$, that is when the $B_{x,t}$ variables are distributed uniformly on $(0,1)$, the expressions for $x(\theta)$ and $I(x(\theta))$ simplify. We find that

$$x(\theta) = \frac{1+2\theta}{\theta^2 + (\theta+1)^2}$$

and

$$I(x(\theta)) = \frac{1}{\theta^2 + (\theta+1)^2},$$

so that the rate function I is simply the function $I : x \mapsto 1 - \sqrt{1-x^2}$.

The following theorem gives a second order correction to the large deviation principle satisfied by the position of the walker at time t.

Theorem 4.1.15. *For $0 < \theta < 1/2$ and $\alpha = \beta = 1$, we have that*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\log\Big(P\big(t, x(\theta)t\big)\Big) + I\big(x(\theta)\big)t}{t^{1/3}\sigma\big(x(\theta)\big)} \leqslant y\right) = F_{\mathrm{GUE}}(y). \tag{4.11}$$

Remark 4.1.16. As we explain in Section 4.5, we expect Theorem 4.1.15 to hold more generally for arbitrary parameters $\alpha, \beta > 0$ and $\theta > 0$. The assumption $\alpha = \beta$ is made for simplifying the computations, whereas the assumption $\theta < 1/2$ is present because certain deformations of contours are justified only for $\theta < \min\{1/2, \alpha + \beta\}$. The condition $\theta > 0$ is natural, it corresponds to looking at $x(\theta) < 1$. We know that for $x(\theta) > 1$, then $P(t, x(\theta)t) = 0$.

In the case $\alpha = \beta = 1$, the condition $\theta < 1/2$ corresponds to $x(\theta) > 4/5$.

Remark 4.1.17. The Tracy-Widom limit theorem from Theroem 4.1.15 should be understood as an analogue of limit theorems for the free energy fluctuations of exactly-solvable random directed polymers. Similar results are proved in [ACQ11, BCF14] for the continuum polymer, in [BC14, BCF14] for the O'Connell-Yor semi-discrete polymer, in [BCR13] for the log-gamma polymer, and in [OO15, CSS15] for the strict-weak-lattice polymer.

In light of KPZ universality for directed polymers, we expect the conclusion of Theorem 4.1.15 to be more general with respect to weight distribution, but this is only the first RWRE to verify this.

In Section 4.5, we also provide an interesting corollary of Theorem 4.1.15. Corollary 4.5.8 states that if one considers an exponential number of Beta RWRE drawn in the same environment, then the maximum of the endpoints satisfies a Tracy-Widom limit theorem. It turns out that even if the rescaled endpoint of a random walk converges in distribution to a Gaussian random variable for large t, the limit theorem that we get is quite different from the one verified by Gaussian random variables having the same dependence structure.

4.1.6 Localization of the paths

The localization properties of random walks in random environment are quite different from localization properties of random directed polymers in 1+1 dimensions. For instance, in the log-gamma polymer model, the endpoint of a polymer path of size n fluctuates on the scale $n^{2/3}$ [Sep12], and localizes in a region of size $\mathcal{O}(1)$ when one conditions on the environment[CN14]. For random walks in random environment, it is clear by the central limit theorem that the endpoint of a path of size n fluctuates on the scale $\sqrt{n}$.

Remarkably, the central limit theorem also holds if one conditions on the environment. A general quenched central limit theorem is proved in [RAS05] for space-time i.i.d. random walks in $\mathbb{Z}^d$. The only hypotheses are that the walk is not deterministic, and that the expectation over the environment of the variance of an elementary increment is finite. These two conditions are clearly satisfied by the Beta-RWRE model. In the particular case of one-dimensional random walks, and when transition probabilities have mean 1/2, the result was also proved in [Bé04]. However, most of the other papers proving a quenched central limit theorem for similar RW models assume a strict ellipticity condition, which is not satisfied by the Beta-RWRE. See also [RAS09, BSS14] for similar results about random walks in random environment under weaker conditions.

In any case, if we let the environment vary, the fluctuations of the endpoints at time t in the Beta RWRE live on the $\sqrt{t}$ scale. For the Beta-RWRE, Proposition 4.5.13 shows that the expected proportion of overlap between two random walks drawn independently in a common environment is of order $\sqrt{t}$ up to time t. The $\sqrt{t}$ order of magnitude has already been proved in [RAS05, Lemma 2] based on results from [FF98], and our Proposition 4.5.13 provides the precise equivalent.

Let us give an intuitive argument explaining the difference of behaviour between polymers and random walks. Assume that the environment of the random walk (resp. the polymer) has been drawn, and consider a random walk starting from the point 0 (resp. a point-to-point polymer starting from 0). The quenched probability that the random walk performs a first step upward depends only on the environment at the point 0 (i.e. the random variable $B_{0,0}$ in the case of the Beta RWRE). However, the probability for the polymer path to start with a step upward depends on the global environment. For instance, if the weight on some edge is very high, this will influence the probability that the first step of the polymer path is upward or downward, so as to enable the polymer path to go through the edge with high weight. This explains why two independent paths in the same environment have more tendency to overlap in the polymer model.

In [GRASY13], a random walk in dynamic random environment is associated to a random directed polymer in 1+1 dimensions, under a condition called north-east induction on the edge-weights. For the log-gamma polymer, it turns out that the associated random walk has Beta distributed transition probabilities. However, the environment is correlated, so that this RWRE is very different from the Beta RWRE. The random walk considered in [GRASY13] defines a measure on lattice paths which can be seen as a limit of point-to-point polymer measures. Hence, as pointed out in [GRASY13, Remark 8.3], it has very different localization properties than random walks in space-time i.i.d random environment that we consider in the present paper.

4.1.7 Limit theorem at zero-temperature

Turning to the zero-temperature limit, Theorem 4.1.13 degenerates to the following for the Bernoulli-Exponential FPP model:

Theorem 4.1.18. *For $r \in \mathbb{R}_{>0}$, fix $n, m \geqslant 0$ and consider $T(n,m)$ the first passage time to the set $D_{n,m}$ in the Bernoulli-Exponential FPP model with parameters $a, b > 0$. Then, one has*

$$\mathbb{P}\Big(T(n,m) > r\Big) = \det(I + K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0')}$$

where C_0' is a small positively oriented circle containing 0 but not $-\nu$, and $K_r^{\mathrm{FPP}} : \mathbb{L}^2(C_0') \to \mathbb{L}^2(C_0')$ is defined by its integral kernel

$$K_r^{\mathrm{FPP}}(u,u') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \frac{e^{rs}}{s}\frac{g^{\mathrm{FPP}}(u)}{g^{\mathrm{FPP}}(u+s)}\frac{\mathrm{d}s}{s+u-u'}, \tag{4.12}$$

where

$$g^{\mathrm{FPP}}(u) = \left(\frac{a+u}{u}\right)^n \left(\frac{a+u}{a+b+u}\right)^m \frac{1}{u}. \tag{4.13}$$

The integral in (4.12) is an improper oscillatory integral if one integrates on the vertical line $1/2 + i\mathbb{R}$. One could justify a deformation of the integration contour (so that the tails go to $\infty e^{\pm i2\pi/3}$ for instance) in order to have an absolutely convergent integral, but it happens that the vertical contour is more practical for analyzing the asymptotic behaviour of $\det(I + K_r^{\mathrm{FPP}})$ in Section 4.6.

One has a Tracy-Widom limit theorem for the fluctuations of the first passage time $T(n,\kappa n)$ when n goes to infinity, for some slope $\kappa > \frac{a}{b}$. Theorem 4.1.19 is proved as Theorem 4.6.1 in Section 4.6.

Theorem 4.1.19. *We have that for any $\theta > 0$ and parameters $a, b > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T\big(n, \kappa(\theta)n\big) - \tau(\theta)n}{\rho(\theta)n^{1/3}} \leqslant y\right) = F_{\mathrm{TW}}(y),$$

where $\kappa(\theta), \tau(\theta)$ and $\rho(\theta)$ are explicit constants (see Section 4.6) such that when θ ranges from 0 to infinity, $\kappa(\theta)$ ranges from $+\infty$ to a/b.

Notice that in Theorem 4.1.19, we do not have any restriction on the range of the parameters a, b and θ.

Another direction of study for the Bernoulli-Exponential FPP model is to compute the asymptotic shape of the percolation cluster $C(t)$ for a fixed time t (but looking very far from the origin). In Section 4.6.3 we explain, based on a degeneration of the results of Theorem 4.1.19, what should be the limit shape of the convex envelope of the percolation cluster, and guess the scale of the fluctuations. However, these arguments are based on a non-rigorous interchange of limits and we leave a rigorous proof for future consideration.

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Outline of the paper

In Section 4.2, we introduce the q-Hahn TASEP [Cor14, Pov13] and show how some observables of the q-Hahn TASEP converge to the partition function of the Beta polymer (and likewise endpoint distribution of the Beta RWRE). This enables us to give a first proof of the Fredholm determinant formulas in Theorems 4.1.12 and 4.1.13. In Section 4.3, we give a direct proof of Theorems 4.1.12 and 4.1.13 using an approach which can be seen as a rigorous instance of the replica method. In Section 4.4, we show that the Beta RWRE converges to the Bernoulli-Exponential FPP, and prove the Fredholm determinant formula of Theorem 4.1.18. In Section 4.5 we perform an asymptotic analysis of the Fredholm determinant from Theorem 4.1.13 to prove Theorem 4.1.15. We also discuss Corollary 4.5.8 which is about the maximum of the endpoints of several Beta RWRE drawn in a common environment, and we relate this result to extreme value theory. In Section 4.6, we perform an asymptotic analysis of the Bernoulli-Exponential FPP model to prove Theorem 4.1.19.

4.2 From the q-Hahn TASEP to the Beta polymer

In this section, we explain how the Beta-RWRE and the Beta polymer arise as limits of the q-Hahn TASEP introduced in [Pov13] (see also [Cor14]). We first show that some observables of the q-Hahn TASEP converge to the partition function of the polymer model

(Proposition 4.2.1). This yields a first proof of Theorem 4.1.12. Then we prove that the Beta RWRE and the Beta polymer model are equivalent models in the sense of Proposition 4.1.6.

4.2.1 The q-Hahn TASEP

Let us recall the definition of the q-Hahn-TASEP: This is a discrete time interacting particle system on the one-dimensional integer lattice. Fix $0 < q < 1$ and $0 \leqslant \bar{\nu} \leqslant \bar{\mu} < 1$. Then the N-particle q-Hahn TASEP is a discrete time Markov chain $\vec{x}(t) = \{x_n(t)\}_{n=0}^N \in \mathbb{X}_N$ where the state space $\mathbb{X}_N$ is

$$\mathbb{X}_N = \Big\{ +\infty = x_0 > x_1 > \dots > x_N : \forall i, x_i \in \mathbb{Z} \Big\}.$$

At time $t+1$, each coordinate $x_n(t)$ is updated independently and in parallel to $x_n(t+1) = x_n(t) + j_n$ where $0 \leqslant j_n \leqslant x_{n-1}(t) - x_n(t) - 1$ is drawn according to the q-Hahn probability distribution $\varphi_{q,\bar{\mu},\bar{\nu}}(j_n | x_{n-1}(t) - x_n(t) - 1)$. The q-Hahn probability distribution on $j \in \{0, 1, \dots, m\}$ is defined by the probabilities

$$\varphi_{q,\bar{\mu},\bar{\nu}}(j|m) = \bar{\mu}^j \frac{(\bar{\nu}/\bar{\mu}; q)_j (\bar{\mu}; q)_{m-j}}{(\bar{\nu}; q)_m} \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}}, \tag{4.14}$$

where for $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$, $(a;q)_n$ is the q-Pochhammer symbol

$$(a;q)_n = (1-a)(1-qa)\dots(1-q^{n-1}a).$$

4.2.2 Convergence of the q-Hahn TASEP to the Beta polymer

An interesting interpretation of the q-Hahn distribution is provided in Section 4 of [GO09]. The authors define a q-analogue of the Pólya urn process: One considers two urns, initially empty, in which one sequentially adds balls. When the first urn contains k balls, and the second urn contains $n-k$ balls, one adds a ball to the first urn with probability $[\nu - \mu + n - k]_q / [\nu + n]_q$, where for any integer m, $[m]_q = (1-q^m)/(1-q)$ denotes the q-deformed integer, and we set $\bar{\mu} = q^{\mu}$ and $\bar{\nu} = q^{\nu}$. One adds a ball to the second urn with the complementary probability. Then $\varphi_{q,\bar{\mu},\bar{\nu}}(j|m)$ is the probability that after m steps, the first urn contains j balls. When q goes to 1, one recovers the classical Pólya urn process.

For the classical Pólya urn, it is known that after n steps, the number of balls in the first urn is distributed according to the Beta-Binomial distribution. Further, the proportion of balls in the first urns converges in distribution to the Beta distribution when the number of added balls tends to infinity. Thus, it is natural to consider the q-Hahn distribution as a q-analogue of the Beta-Binomial distribution. Further, we expect that if X is a random variable drawn according to the q-Hahn distribution on $\{0, \dots, m\}$ with parameters $(q, \bar{\mu}, \bar{\nu})$, the q-deformed proportion $[X]_q/[m]_q$ converges as m goes to infinity to a q analogue of the Beta distribution, which converges as q goes to 1 to the Beta distribution with parameters $(\nu - \mu, \mu)$.

Now, we show that the partition function of the Beta polymer is a limit of observables of the q-Hahn TASEP. Let $F^{\epsilon}(t,n)$ be the rescaled quantity

$$F^{\epsilon}(t,n) = -\epsilon(x_n(t) + n), \tag{4.15}$$

where $x_n(t)$ is the location of the nth particle in q-Hahn TASEP and we set $q = e^{-\epsilon}, \bar{\mu} = q^{\mu}$ and $\bar{\nu} = q^{\nu}$.

Proposition 4.2.1. *For $t \geqslant 0$ and $n \geqslant 1$ such that $n \leqslant t+1$, the sequence of random variables $(F^\epsilon(t,n))_\epsilon$ converges in distribution as $\epsilon \to 0$ to a limit $F(t,n)$ and one has*

$$e^{F(t,n)} = e^{F(t-1,n)} B_{t,n} + e^{F(t-1,n-1)}(1-B_{t,n})$$

where $B_{t,n}$ are i.i.d. Beta distributed random variables with parameters $(\mu, \nu-\mu)$. Additionally, we have the weak convergence of processes

$$\{e^{F^\epsilon(t,n)}\}_{t\geqslant 0, n\geqslant 1} \Rightarrow \{Z(t,n)\}_{t\geqslant 0, n\geqslant 1}.$$

Proof. We first state a lemma useful for taking limits of q-Pochhammer symbols.

Lemma 4.2.2. *For $r, q \in (0,1)$ and $x > 0$, we have that*

$$\frac{(r;q)_\infty}{(rq^x;q)_\infty} \xrightarrow[q\to 1]{} (1-r)^x.$$

Proof. We take the limit of $\log\left(\frac{(r;q)_\infty}{(rq^x;q)_\infty}\right)$. Since $r, q \in (0,1)$, one can use the series expansion of the logarithm around 1. This yields

$$\begin{aligned}
\log\left(\frac{(r;q)_\infty}{(rq^x;q)_\infty}\right) &= \sum_{i=0}^{\infty} \log\left(\frac{1-rq^i}{1-rq^{x+i}}\right) \\
&= \sum_{i=0}^{\infty} -\left(\sum_{j=1}^{\infty}\left(\frac{(rq^i)^j}{j} - \frac{(rq^{x+j})^j}{j}\right)\right) \\
&= \sum_{j=1}^{\infty} \frac{-r^j}{j}\left(\sum_{i=0}^{\infty}(q^j)^i - q^{xj}\sum_{i=0}^{\infty}(q^j)^i\right) \\
&= -\sum_{j=1}^{\infty} \frac{r^j}{j}\frac{1-q^{xj}}{1-q^j} \\
&\xrightarrow[q\to 1]{} x\log(1-r).
\end{aligned}$$

The last convergence comes from term-wise convergence as $q \to 1$ along with absolute convergence of the sum for $r \in (0,1)$. ☐

Lemma 4.2.3. *The sequence of random variables $\exp(F^\epsilon(1,1))$ converges as $\epsilon \to 0$ to a Beta distributed random variable with parameters $(\mu, \nu-\mu)$.*

Proof. By the definition of $F^\epsilon(t,n)$ given in Equation (4.15),

$$\exp(F^\epsilon(1,1)) = r \iff x_1(1) + 1 = -\epsilon^{-1}\log r.$$

For r such that $-\epsilon^{-1}\log(r) \in \mathbb{Z}$,

$$\begin{aligned}
\mathbb{P}\left(x_1(1)+1 = -\epsilon^{-1}\log r\right) &= \varphi_{q,\bar\mu,\bar\nu}(-\epsilon^{-1}\log r|\infty) \\
&= \bar\mu^{-\epsilon^{-1}\log r}\frac{(\bar\nu/\bar\mu;q)_{-\epsilon^{-1}\log r}}{(q;q)_{-\epsilon^{-1}\log r}}\frac{(\bar\mu;q)_\infty}{(\bar\nu;q)_\infty}.
\end{aligned}$$

We shall use the q-Gamma function defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-z}.$$

Note that for $j \in \mathbb{Z}_{\geqslant 0}$,

$$(q^x;q)_j = \frac{(q^x;q)_\infty}{(q^{x+j};q)_\infty} = (1-q)^j \frac{\Gamma_q(x+j)}{\Gamma_q(x)},$$

so that

$$\begin{aligned}\mathbb{P}\left(X_1(1)+1 = -\epsilon^{-1}\log r\right) &= r^\mu \frac{\frac{(q^{\nu-\mu};q)_\infty}{(rq^{\nu-\mu};q)_\infty}}{\frac{(q;q)_\infty}{(rq;q)_\infty}} \frac{(1-q)^{\nu-\mu}\Gamma_q(\nu)}{\Gamma_q(\mu)} \\ &= (1-q)r^\mu \frac{(rq;q)_\infty}{(rq^{\nu-\mu};q)_\infty} \frac{\Gamma_q(\nu)}{\Gamma_q(\nu-\mu)\Gamma_q(\mu)}.\end{aligned}$$

As ϵ goes to zero, using Lemma 4.2.2 and the fact that $\Gamma_q(x) \xrightarrow[q\to 1]{} \Gamma(x)$,

$$\frac{\Gamma_q(\nu)}{\Gamma_q(\nu-\mu)\Gamma_q(\mu)} \to \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)};$$

$$\frac{(rq;q)_\infty}{(rq^{\nu-\mu};q)_\infty} \to (1-r)^{\nu-\mu-1}.$$

Thus as ϵ goes to zero,

$$\epsilon^{-1}\mathbb{P}\left(F^\epsilon(1,1) = \log r\right) \to r \times r^{\mu-1}(1-r)^{\nu-\mu-1}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}. \tag{4.16}$$

Hence $F^\epsilon(1,1)$, which takes values in $a_\epsilon + \epsilon\mathbb{Z}$ where a_ϵ is an ϵ-dependent shift, converges weakly to a continuous random variable F whose density is given by $f(s) = \lim \epsilon^{-1}\mathbb{P}\left(F^\epsilon(1,1) = s_\epsilon\right)$ (where s_ϵ is the closest point to s in $a_\epsilon + \epsilon\mathbb{Z}$). For more details, see the proof of [CSS15, Lemma 2.1] which is very similar. Consequently, $\exp(F^\epsilon(1,1))$ converges weakly to the continuous random variable $\exp(F)$. Since the density of F is given by the right-hand-side of (4.16) with $s = \log r$, one concludes that the density of $\exp(F(1,1))$ is

$$r^{\mu-1}(1-r)^{\nu-\mu-1}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}$$

which is the density of a $Beta(\mu, \nu-\mu)$ random variable.

□

Lemma 4.2.4. *Conditionally on $e^{F^\epsilon(t-1,n)} = Z$ and $e^{F^\epsilon(t-1,n-1)} = Z'$, the sequence of random variables $\frac{\exp(F^\epsilon(t,n))-Z'}{Z-Z'}$ converges as $\epsilon \to 0$ to a Beta distributed random variable with parameters $(\mu, \nu-\mu)$.*

Proof. Conditioning on $e^{F^\epsilon(t-1,n)} = Z$ and $e^{F^\epsilon(t-1,n-1)} = Z'$ corresponds to conditioning on the fact that the gap $x_{n-1}(t-1) - x_n(t-1) - 1$ is $\epsilon^{-1}\log(Z/Z')$. The probability that $e^{F^\epsilon(t,n)}/Z = s$, conditioned to $e^{F^\epsilon(t-1,n)} = Z$ and $e^{F^\epsilon(t-1,n-1)} = Z'$ is

$$\mathbb{P}\Big(x_n(t) - x_n(t-1) = -\epsilon^{-1}\log(s)\Big| x_{n-1}(t-1) - x_n(t-1) - 1 = -\epsilon^{-1}\log(r)\Big),$$

where we have set $r = Z'/Z$. By the definition of the q-Hahn TASEP, this probability is exactly $\varphi_{q,\bar\mu|\bar\nu}\big(-\epsilon^{-1}\log(s)\big| -\epsilon^{-1}\log(r)\big)$.

$$\begin{aligned}\varphi_{q,\bar\mu,\bar\nu}\Big(-\epsilon^{-1}\log(s)\Big| -\epsilon^{-1}\log(r)\Big) = &\\ s^\mu \frac{(q^\mu;q)_{-\epsilon\log(r/s)}}{(q^\nu;q)_{-\epsilon\log(r)}}\frac{(q^{\nu-\mu};q)_{-\epsilon\log(s)}}{(q;q)_{-\epsilon\log(s)}}\frac{(q;q)_{-\epsilon\log(r)}}{(q;q)_{-\epsilon\log(r/s)}}&\\ = (1-q)s^\mu\frac{(\frac{r}{s}q^\nu;q)_\infty}{(\frac{r}{s}q^\mu;q)_\infty}\frac{(sq;q)_\infty}{(sq^{\nu-\mu};q)_\infty}\frac{(rq;q)_\infty}{(\frac{r}{s}q;q)_\infty}\frac{\Gamma_q(\nu)}{\Gamma_q(\nu-\mu)\Gamma_q(\mu)}&.\end{aligned}$$

Using again Lemma (4.2.2), one finds that as ϵ goes to zero,

$$\epsilon^{-1}\varphi_{q,\bar\mu,\bar\nu}\Big(-\epsilon^{-1}\log(s)\Big| -\epsilon^{-1}\log(r)\Big) \to s^\mu(1-s)^{\nu-\mu-1}\frac{(1-r/s)^{\mu-1}}{(1-r)^{\nu-1}}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}$$

which can be rewritten as

$$\begin{aligned}\epsilon^{-1}\varphi_{q,\bar\mu,\bar\nu}\Big(-\epsilon^{-1}\log(s)\Big| -\epsilon^{-1}\log(r)\Big) \to &\\ \frac{s}{1-r}\times\left(\frac{s-r}{1-r}\right)^{\mu-1}\left(\frac{1-s}{1-r}\right)^{\nu-\mu-1}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}&. \qquad (4.17)\end{aligned}$$

By the same arguments as in the proof of Lemma 4.2.3, this shows that $\exp F^\epsilon(t,n)/Z$ converges to a continuous random variable in $(r,1)$ having density

$$\frac{1}{1-r}\left(\frac{s-r}{1-r}\right)^{\mu-1}\left(\frac{1-s}{1-r}\right)^{\nu-\mu-1}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)\Gamma(\mu)}\mathrm{d}s.$$

This implies that

$$\frac{\exp F^\epsilon(t,n)/Z - Z'/Z}{1-Z'/Z} \Longrightarrow Beta(\mu,\nu-\mu).$$

□

It is clear that $\exp F^\epsilon(0,1) = 1$ for any ϵ. By applying several times Lemma 4.2.3, one sees that $\exp F^\epsilon(t,1)$ converges to a product of independent Beta random variables with parameters $(\mu,\nu-\mu)$. Finally, by recurrence on $t+n$ and using Lemma 4.2.4, $\exp F^\epsilon(t,n)$ converges in distribution to $\exp F(t,n)$ where the family of random variables $(\exp F(t,n))_{t,n}$ is defined by the recurrence formula of the statement of Proposition 4.2.1. This, in turn, implies the weak convergence of processes

$$\{e^{F^\epsilon(t,n)}\}_{t\geqslant 0,n\geqslant 1} \Rightarrow \{Z(t,n)\}_{t\geqslant 0,n\geqslant 1}. \qquad (4.18)$$

□

One has the following Fredholm determinant representation for the e_q-Laplace transform of $X_n(t)$.

Theorem 4.2.5 (Theorem 1.10 in [Cor14])**.** *Consider q-Hahn TASEP started from step initial data $x_n(0) = -n$ $\forall n \geqslant 1$. Then for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,*

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right] = \det(I + K_\zeta^{\mathrm{qHahn}})_{\mathbb{L}^2(C_1)} \tag{4.19}$$

where C_1 is a small positively oriented circle containing 1 but not $1/\bar{\nu}, 1/q$ nor 0, and $K_\zeta^{\mathrm{qHahn}} : \mathbb{L}^2(C_1) \to \mathbb{L}^2(C_1)$ is defined by its integral kernel

$$K_\zeta^{\mathrm{qHahn}}(w, w') = \frac{1}{2i\pi} \int_{1/2+i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-\zeta)^s \frac{g^{\mathrm{qHahn}}(w)}{g^{\mathrm{qHahn}}(q^s w)} \frac{\mathrm{d}s}{q^s w - w'}$$

with

$$g(w) = \left(\frac{(\bar{\nu} w; q)_\infty}{(w; q)_\infty}\right)^n \left(\frac{(\bar{\mu} w; q)_\infty}{(\bar{\nu} w; q)_\infty}\right)^t \frac{1}{(\bar{\nu} w; q)_\infty}.$$

Let us scale the parameter ζ as

$$\zeta = (1-q)u,$$

and scale the other parameters as previously: $q = e^{-\epsilon}, \bar{\mu} = q^\mu, \bar{\nu} = q^\nu$. Then we have

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right] = \mathbb{E}\left[e_q\left(ue^{F^\epsilon(t,n)}\right)\right]$$

where

$$e_q(x) = \frac{1}{\big((1-q)x; q\big)_\infty}$$

is the e_q-exponential function. Since $e_q(x) \to e^x$ uniformly for x in a compact set, we have, using the convergence of processes (4.18) and the fact that $e^{F^\epsilon(t,n)}$ are uniformly bounded by 1, that

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right] = \mathbb{E}\left[\exp(ue^{F(t,n)})\right]. \tag{4.20}$$

Hence, in order to prove Theorem 4.1.12, one has to take the limit when ϵ goes to zero of the Fredholm determinant in the right-hand-side of (4.19). This achieved in Proposition 4.2.6.

Proposition 4.2.6.

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n}; q)_\infty}\right] = \det(I + K_u^{\mathrm{BP}})_{\mathbb{L}^2(C_0)}$$

where C_0 is a small positively oriented circle containing 0 but not $-\nu$ nor -1, and $K_u^{\mathrm{BP}} : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

$$K_u^{\mathrm{BP}}(v, v') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+s)} \frac{\mathrm{d}s}{s + v - v'} \tag{4.21}$$

where

$$g^{\mathrm{BP}}(v) = \left(\frac{\Gamma(v)}{\Gamma(\nu+v)}\right)^n \left(\frac{\Gamma(\nu+v)}{\Gamma(\mu+v)}\right)^t \Gamma(\nu+v).$$

Proof. Let us first show that the pointwise limit of the kernel of the Fredholm determinant (4.19) of the q-Hahn TASEP agrees with (4.21). Make the change of variables

$$w = q^v, \quad w' = q^{v'}.$$

The function g^{qHahn} used inside the integrand of the kernel K_ζ^{qHahn} becomes

$$g^{\mathrm{qHahn}}(q^v) = \left(\frac{(q^{\nu+v};q)_\infty}{(q^v;q)_\infty}\right)^n \left(\frac{(q^{\mu+v};q)_\infty}{(q^{\nu+v};q)_\infty}\right)^t \frac{1}{(q^{\nu+v};q)_\infty}.$$

We again use the q-Gamma function and the formula

$$(q^z;q)_\infty = \frac{(q;q)_\infty}{\Gamma_q(z)}(1-q)^{1-z}.$$

In terms of q-Gamma function

$$g^{\mathrm{qHahn}}(q^v) = \left(\frac{(1-q)^v\Gamma_q(v)}{(1-q)^{\nu+v}\Gamma_q(\nu+v)}\right)^n \left(\frac{(1-q)^{\nu+v}\Gamma_q(\nu+v)}{(1-q)^{\mu+v}\Gamma_q(\mu+v)}\right)^t \frac{(1-q)^{\nu+v}\Gamma_q(\nu+v)}{(1-q)(q;q)_\infty}.$$

Thus,

$$\frac{g^{\mathrm{qHahn}}(q^v)}{g^{\mathrm{qHahn}}(q^{v+s})} = \left(\frac{\Gamma_q(\nu+v+s)\Gamma_q(v)}{\Gamma_q(v+s)\Gamma_q(\nu+v)}\right)^n \left(\frac{\Gamma_q(\mu+v+s)\Gamma_q(\nu+v)}{\Gamma_q(\nu+v+s)\Gamma_q(\mu+v)}\right)^t \frac{\Gamma_q(\nu+v)}{(1-q)^s\Gamma_q(\nu+v+s)}.$$

and hence,

$$(1-q)^s \frac{g^{\mathrm{qHahn}}(q^v)}{g^{\mathrm{qHahn}}(q^{v+s})} \xrightarrow[\epsilon\to 0]{} \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+s)}.$$

The extra factor $(1-q)^s$ in $g(q^v)/g(q^{v+s})$ cancels with the one coming from ζ. Moreover,

$$\frac{-\epsilon}{q^{s+v}-q^{v'}} \xrightarrow[\epsilon\to 0]{} \frac{1}{s+v-v'},$$

where the factor $-\epsilon$ comes from the Jacobian of the change of variables $\mathrm{d}w = -\epsilon q^v \mathrm{d}v$. This demonstrates pointwise convergence of the integrand in the integral defining K_ζ^{qHahn} to that defining K_u^{BP}.

In order to show that $-\epsilon K_\zeta^{\mathrm{qHahn}}(q^v, q^{v'})$ converges to $K_u^{\mathrm{BP}}(v, v')$, one needs an integrable bound. We will to show that for a fixed v, the quantity

$$\frac{\pi}{\sin(\pi s)} \frac{\Gamma_q(\mu+v+s)^t}{\Gamma_q(\nu+v+s)^{t-n}\Gamma_q(v+s)^n} \frac{1}{\Gamma_q(\nu+v+s)} \tag{4.22}$$

is uniformly integrable in s as q varies. We need a few estimates to show this. For $z = x+iy$ with fixed x, we have from [EMO+53, Chapter 1, 1.18 (2)],

$$\big|\Gamma(x+iy)\big|e^{|y|\pi/2}|y|^{1/2-x} \xrightarrow[|y|\to\infty]{} e^{-x}\sqrt{2\pi}. \tag{4.23}$$

We also need the estimates for the q-Gamma function in the two next lemmas.

Lemma 4.2.7. *For any fixed $a, b > 0$, there exists a constant $C_2 > 0$ such that for any $y \in \mathbb{R}$ and $q \in (\frac{1}{2}, 1)$,*

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q(b+iy)}\right| \leqslant C_2\Big(\big|y\big|^{|b-a|+1} + 1\Big).$$

Proof. By symmetry, it is enough to prove the result for $y > 0$. We have

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q(b+iy)}\right| = (1-q)^{b-a}\left|\prod_{n\geqslant 0}\left(\frac{1-q^{b+iy+n}}{1-q^{a+iy+n}}\right)\right|.$$

If $a < b$, then

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q(b+iy)}\right| \leqslant (1-q)^{b-a}\left|\prod_{n\geqslant 0}\left(\frac{1-q^{b+n}}{1-q^{a+n}}\right)\right| = \frac{\Gamma_q(a)}{\Gamma_q(b)} \leqslant 1.$$

If $a > b$, We write

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q(b+iy)}\right| = \left|\frac{\Gamma_q(a+iy)}{\Gamma_q\big(b+\lceil a-b\rceil+iy\big)}\frac{\Gamma_q\big(b+\lceil a-b\rceil+iy\big)}{\Gamma_q(b+iy)}\right|.$$

Since $b+\lceil a-b\rceil > a$, we have from the first part of the proof that

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q\big(b+\lceil a-b\rceil+iy\big)}\right| \leqslant 1.$$

Moreover, since the q-Gamma function satisfies the functional equation

$$\Gamma_q(z+1) = [z]_q\Gamma_q(z),$$

where $[z]_q := \frac{1-q^z}{1-q}$ is the q-deformed complex number, we have that

$$\left|\frac{\Gamma_q(a+iy)}{\Gamma_q(b+iy)}\right| \leqslant \prod_{j=0}^{\lceil a-b\rceil-1}\big|[b+j+iy]_q\big|.$$

It can be checked that for $x, y \in \mathbb{R}$, we have the identity

$$\big|[x+iy]_q\big|^2 = \big|[x]_q\big|^2 + q^x\big|[iy]_q\big|^2.$$

For $q \in (1/2, 1)$, $|\log(q)/(1-q)|^2 \leqslant 2$, and hence

$$\big|[iy]_q\big|^2 = 2\frac{\Big(1-\cos\big(y\log(q)\big)\Big)}{(1-q)^2} \leqslant \frac{y^2\log(q)^2}{(1-q)^2} \leqslant 2y^2.$$

This implies that there exist a constant $C_2 > 0$ independent of q such that

$$\left|\prod_{j=0}^{\lceil a-b\rceil-1}[b+j+iy]_q\right| \leqslant C_2(y^{(a-b+1)}+1),$$

which concludes the proof. □

Lemma 4.2.8. *For any $y \in \mathbb{R}$ and $q \in (0,1)$,*

$$\big|\Gamma(1+iy)\big| \leqslant \big|\Gamma_q(1+iy)\big|.$$

Proof. We have

$$\Gamma_q(1+iy) = (1-q)^{-iy}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{n+iy}},$$

so that

$$\big|\Gamma_q(1+iy)\big| = \prod_{n=1}^{\infty}\frac{1-q^n}{\sqrt{1+q^{2n}-2q^n\cos\big(y\log(q)\big)}}.$$

We also have that

$$\Gamma(1+iy) = \prod_{n=1}^{\infty}\frac{n}{n+iy}\left(\frac{n+1}{n}\right)^{iy},$$

so that

$$\big|\Gamma(1+iy)\big| = \prod_{n=1}^{\infty}\left|\frac{n}{n+iy}\right| = \prod_{n=1}^{\infty}\frac{n}{\sqrt{n^2+y^2}}.$$

Hence, it is enough to show that for all $n \geqslant 1$,

$$\frac{n}{\sqrt{n^2+y^2}} \leqslant \frac{1-q^n}{\sqrt{1+q^{2n}-2q^n\cos\big(y\log(q)\big)}}.$$

Setting $Y = y/n$ and $Q = q^n$, it is equivalent to

$$\frac{1}{1+Y^2} \leqslant \frac{(1-Q)^2}{1+Q^2-2Q\cos\big(Y\log(Q)\big)},$$

which is equivalent to

$$Y^2(1-Q)^2 \geqslant 2Q\Big(1-\cos\big(Y\log(Q)\big)\Big),$$

which is true for any $Q \in (0,1)$ and $Y \in \mathbb{R}$. $\square$

Finally, we can write using Lemma 4.2.8 that for $s \in 1/2 + i\mathbb{R}$,

$$\begin{aligned}\left|\Gamma(s)\Gamma(1-s)\frac{1}{\Gamma_q(\nu+v+s)}\right| &= \left|\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1/2+s)}\frac{\Gamma(1/2+s)}{\Gamma_q(1/2+s)}\frac{\Gamma_q(1/2+s)}{\Gamma_q(\nu+v+s)}\right| \\ &\leqslant \left|\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1/2+s)}\frac{\Gamma_q(1/2+s)}{\Gamma_q(\nu+v+s)}\right|.\end{aligned}$$

Hence, using Lemma 4.2.7, there exist constants $C, c > 0$ such that (4.22) is uniformly bounded by

$$Ce^{-\pi/2|\mathfrak{Im}[s]|}\big|\mathfrak{Im}[s]\big|^c.$$

Thus, (4.22) is uniformly integrable for s along $1/2 + i\mathbb{R}$, as q varies near 1. Consequently the integrand of $-\epsilon K_\zeta^{\mathrm{qHahn}}(q^v, q^{v'})$ is uniformly integrable. By dominated convergence, it implies that we have pointwise convergence of the kernel $-\epsilon K_\zeta^{\mathrm{qHahn}}(q^v, q^{v'})$ to the kernel $K_u^{\mathrm{BP}}(v, v')$.

However, it is a priori not sufficient. In order to prove the convergence of the Fredholm determinant, we use again dominated convergence. First notice that since the Fredholm determinant contour is finite, one can prove as in [CSS15, Lemma 3.2] that $-\epsilon K_\zeta^{\mathrm{qHahn}}(q^v, q^{v'})$

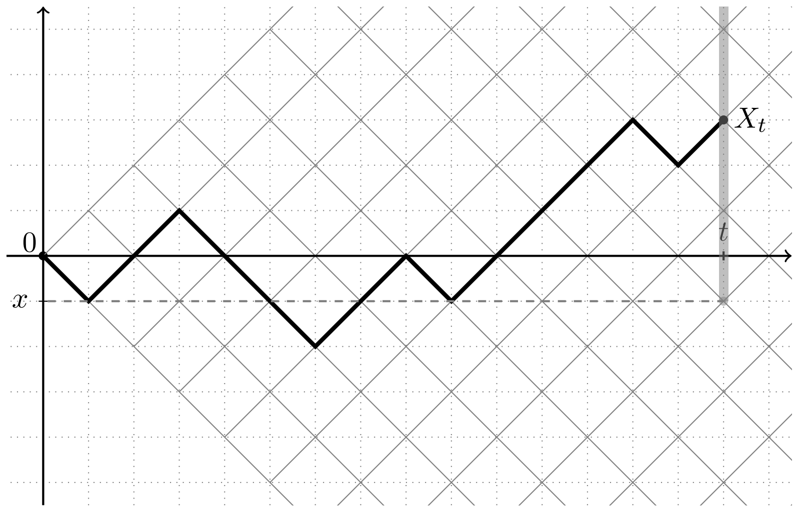

Figure 4.5: A possible path for the Beta-RWRE is shown. It corresponds to the half-line to point polymer path in Figure 4.2. $P(t, x)$ is the (quenched) probability that the random walk ends at time t in the gray region.

is uniformly bounded for v, v' in the contour C_0 and q near 1. Moreover, each term in the Fredholm determinant expansion

$$\det(I + K_\zeta^{\mathrm{qHahn}}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \dots \int \det(K_\zeta^{\mathrm{qHahn}}(w_i, w_j))_{i,j=1}^n \mathrm{d}w_1 \dots \mathrm{d}w_n,$$

can be bounded using Hadamard's bound, so that the sum absolutely converges. Combining this with the above established pointwise convergence of the kernels allows us to conclude the proof of Proposition 4.2.6. □

Proof of Theorems 4.1.12 and 4.1.13. Proposition 4.2.6 combined with (4.20) yields the Fredholm determinant formula for the Laplace transform of $Z(t, n)$ given in Theorem 4.1.12. In order to deduce Theorem 4.1.13, we use the equivalence between the Beta polymer and the Beta-RWRE from Proposition 4.1.6, proved in Section 4.2.3. □

4.2.3 Equivalence Beta-RWRE and Beta polymer

We show that the Beta RWRE and the Beta polymer are equivalent models in the sense that if the parameters α, β of the random walk and the parameters μ, ν of the polymer are such that $\mu = \alpha$ and $\nu = \alpha + \beta$, we have the equality in law

$$Z(t, n) = P(t, t - 2n + 2).$$

The equality in law is true for fixed t and n. However, as families of random variables, $(Z(t, n))$ and $(P(t, t - 2n + 2))$ for $t + 1 \geqslant n \geqslant 1$ have different laws.

Proof of Proposition 4.1.6. Let us first notice that since $\mu = \alpha$ and $\nu = \alpha + \beta$, the i.i.d. collection of Beta random variables defining the environment for the Beta polymer, and

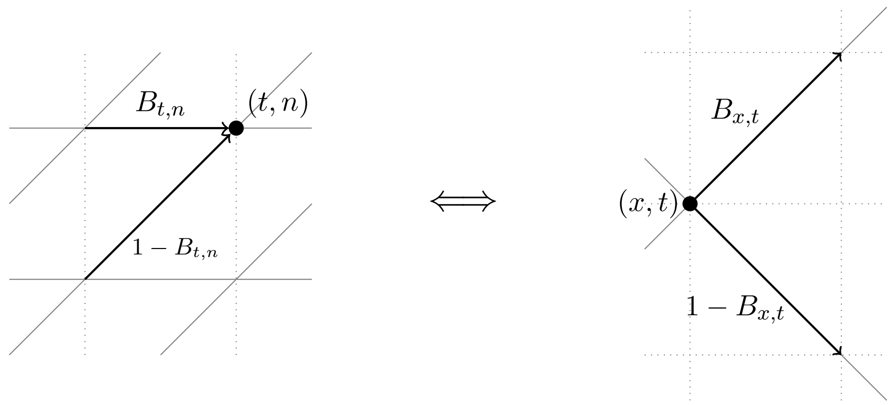

Figure 4.6: Illustration of the deformation of the underlying lattice for the Beta polymer. The left picture corresponds to the Beta polymer whereas the right picture corresponds to the RWRE. Black arrows represents possible steps for the polymer path (resp. the RWRE) with their associated weight (resp. probability).

the i.i.d. collection of r.v. defining the environment of the Beta RWRE, have the same law.

Also, as it was already pointed-out in Section 4.1.2, the point-to-point Beta polymer is equivalent to a half line to point Beta polymer.

Let t and n having the same parity. The random variable $P(t, t-2n+2)$ is the probability for the Beta RWRE to arrive above (or exactly at) $t-2n+2$. This is also the probability for the Beta RWRE to make at most $n-1$ downward steps up to time t. Let us imagine that we deform the underlying lattice of the Beta polymer so that Beta polymer paths are actually up-right path, and we also consider the path from (t,n) to its initial point. Then the polymer path is the trajectory of a random walk, and one can interpret the weight of this polymer path as the quenched probability of the corresponding random walk trajectory (compare the polymer path depicted in Figure 4.2 with the RWRE path depicted in Figure 4.5, using the correspondence shown in Figure 4.6). Moreover the event that the random walks performs at most $n-1$ downward steps is equivalent to the fact that the polymer path starts with positive n-coordinate. These events corresponds to the fact that the path intersects the thick gray half-lines in Figures 4.2 and 4.5.

Finally, for any fixed $t, n \in \mathbb{Z}_{\geqslant 0}$ such that $t+1 \geqslant n$, if we set $x = t-2n+2$, then $P(t,x)$ and $Z(t,n)$ have the same probability law. Moreover, conditioning on the environment of the Beta polymer corresponds to conditioning on the probability of each step for the Beta RWRE. $\square$

4.3 Rigorous replica method for the Beta polymer

4.3.1 Moment formulas

Let $\mathbb{W}^k$ be the Weyl chamber

$$\mathbb{W}^k = \left\{ \vec{n} \in \mathbb{Z}^k : n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \right\}.$$

For $\vec{n} \in \mathbb{W}^k$, let us define

$$u(t, \vec{n}) = \mathbb{E}\big[Z(t, n_1) \dots Z(t, n_k)\big]. \tag{4.24}$$

The recurrence relation (4.1) implies a recurrence relation for $u(t, \vec{n})$. We are going to solve this recurrence to find a closed formula for $u(t, \vec{n})$, using a variant of the Bethe ansatz. It is the analogue of Section 5 in [CSS15]. Besides the strict weak polymer [CSS15], such "replica method" calculations have been performed to study moments of the partition function for the continuum polymer [Dot10, CDR10, BC14], the semi-discrete polymer [BCS14, BC14], and the log-gamma polymer [BC14, TLD14]. However, in those models, the moment problems are ill-posed and one cannot rigorously recover the distribution from them. In the present case, since the $Z(t, n) \in [0, 1]$, the moments do determine the distribution as explained in Section 4.3.2.

Using the recurrence relation (4.1),

$$u(t+1, \vec{n}) = \mathbb{E}\left[\prod_{i=1}^{k} \Big((1 - B_{t+1,n_i})Z(t, n_i) + B_{t+1,n_i} Z(t, n_i - 1)\Big)\right]. \tag{4.25}$$

Let us first simplify this expression when $k = c$ and $\vec{n} = (n, \dots, n)$ is a vector of length c with all components equal. In this case, setting $B = B_{t+1,n_i}$ to simplify the notations, we have

$$\begin{aligned} u(t+1, \vec{n}) &= \sum_{j=0}^{c} \binom{c}{j} \mathbb{E}\left[(1-B)^j B^{c-j} Z(t, n_i - 1)^j Z(t, n_i)^{c-j}\right] \\ &= \sum_{j=0}^{c} \binom{c}{j} \mathbb{E}\left[(1-B)^j B^{c-j}\right] u(t, n, \dots, n, \underbrace{n-1, \dots, n-1}_{j}). \end{aligned}$$

The recurrence relation can be further simplified using the next Lemma.

Lemma 4.3.1. *Let B a random variable following the Beta$(\mu, \nu - \mu)$ distribution. Then for integers $0 \leqslant j \leqslant c$,*

$$\mathbb{E}\big[(1-B)^j B^{c-j}\big] = \frac{(\nu - \mu)_j (\mu)_{c-j}}{(\nu)_c}.$$

where $(a)_k$ is the Pochhammer symbol $(a)_k = a(a+1)\dots(a+k-1)$.

Proof. By the definition of the Beta law, we have

$$\begin{aligned} \mathbb{E}\big[(1-B)^j B^{c-j}\big] &= \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^1 (1-x)^j x^{c-j} x^{\mu-1} (1-x)^{\nu-\mu-1}, \\ &= \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \frac{\Gamma(\mu + c - j)\Gamma(\nu - \mu + j)}{\Gamma(\nu + c)}, \\ &= \frac{(\nu-\mu)_j (\mu)_{c-j}}{(\nu)_c}. \end{aligned}$$

□

In order to write the general case, we need a little more notation. For $\vec{n} \in \mathbb{W}^k$, we denote by $c_1, c_2, \dots c_\ell$ the sizes of clusters of equal components in $\vec{n}$. More precisely, $c_1, c_2, \dots c_\ell$ are positive integers such that $\sum c_i = k$ and

$$n_1 = \dots = n_{c_1} > n_{c_1+1} = \dots n_{c_1+c_2} > \dots > n_{c_1+\dots+c_{k-1}+1} = \dots = n_k.$$

Define also the operator $\tau^{(i)}$ acting on a function $f : \mathbb{W}^k \to \mathbb{R}$ by

$$\tau^{(i)} f(\vec{n}) = f(n_1, \dots, n_i - 1, \dots, n_k).$$

Using the Lemma 4.3.1, we have that

$$u(t+1, \vec{n}) = \sum_{j_1=0}^{c_1} \dots \sum_{j_\ell=0}^{c_\ell} \left(\prod_{i=1}^{\ell} \binom{c_i}{j_i} \frac{(\nu-\mu)_{j_i} (\mu)_{c_i - j_i}}{(\nu)_{c_i}} \prod_{r=0}^{j_i - 1} \tau^{(c_1 + \dots + c_i - r)} \right) u(t, \vec{n}). \quad (4.26)$$

In words, for each ℓ-tuple $j_1, \dots, j_\ell$ such that $0 \leqslant j_i \leqslant c_i$, we decrease the j_i last coordinates of the cluster i in $\vec{n}$, for each cluster, and multiply by

$$\prod_{i=1}^{\ell} \binom{c_i}{j_i} \frac{(\nu-\mu)_{j_i} (\mu)_{c_i - j_i}}{(\nu)_{c_i}}.$$

Lemma 4.3.2. *Let X, Y generate an associative algebra such that*

$$YX = \frac{1}{1+\nu} XX + \frac{\nu - 1}{1+\nu} XY \frac{1}{1+\nu} YY.$$

Then we have the following non-commutative binomial identity:

$$\big(pX + (1-p)Y\big)^n = \sum_{j=0}^{n} \binom{n}{j} \frac{(\nu-\mu)_j (\mu)_{n-j}}{(\nu)_n} X^j Y^{n-j},$$

where $p = \frac{\nu - \mu}{\nu}$.

Proof. It is shown in [Pov13, Theorem 1] that if X and Y satisfy the quadratic homogeneous relation

$$YX = \alpha XX + \beta XY + \gamma YY,$$

with

$$\alpha = \frac{\bar{\nu}(1-q)}{1 - q\bar{\nu}}, \quad \beta = \frac{q - \bar{\nu}}{1 - q\bar{\nu}}, \quad \gamma = \frac{1-q}{1-q\bar{\nu}},$$

and

$$\bar{\mu} = \bar{p} + \bar{\nu}(1 - \bar{p}),$$

then

$$\big(\bar{p}X + (1-\bar{p})Y\big)^n = \sum_{k=0}^{n} \varphi_{q,\bar{\mu},\bar{\nu}}(j|n) X^k Y^{n-k},$$

where $\varphi_{q,\bar{\mu},\bar{\nu}}(j|n)$ are the q-Hahn weights defined in (4.14). Our lemma is the $q \to 1$ degeneration of this result. $\square$

Let us denote $\mathcal{L}_c^{\text{cluster}}$ the operator

$$\mathcal{L}_c^{\text{cluster}} = \sum_{j=0}^{c} \binom{c}{j} \frac{(\nu-\mu)_j (\mu)_{c-j}}{(\nu)_c} \prod_{r=0}^{j-1} \tau^{(c-r)}$$

which appears in the R.H.S. of (4.26), and $\mathcal{L}_c^{free}$ the operator

$$\mathcal{L}_c^{free} = \prod_{i=1}^{c} \nabla_i,$$

where $\nabla_i = p\tau^{(i)} + (1-p)$.

For a function $f : \mathbb{Z}^c \to \mathbb{C}$, we formally identify monomials $X_1X_2\dots X_c$ where $X_i \in \{X, Y\}$ with terms $f(\vec{n})$ where for all $1 \leqslant i \leqslant c$, $n_{c-i} = n-1$ if $X_i = X$ and $n_{c-i} = n$ if $X_i = Y$. Using this identification, the binomial formula from Lemma 4.3.2 says that the operators $\mathcal{L}_c^{free}$ and $\mathcal{L}_c^{\mathrm{cluster}}$ act identically on functions f satisfying the condition

$$\left(\frac{1}{1+\nu}\tau^{(i)}\tau^{(i+1)} + \frac{\nu-1}{1+\nu}\tau^{(i+1)} + \frac{1}{1+\nu} - \tau^{(i)}\right) f(n,\dots,n) = 0. \tag{4.27}$$

One notices that the operator involved in (4.26) acts independently by $\mathcal{L}_c^{\mathrm{cluster}}$ on each cluster of equal components. It follows that if a function $u : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}^k \to \mathbb{C}$ satisfies the *boundary condition*

$$\left(\frac{1}{1+\nu}\tau^{(i)}\tau^{(i+1)} + \frac{\nu-1}{1+\nu}\tau^{(i+1)} + \frac{1}{1+\nu} - \tau^{(i)}\right) u(t,\vec{n}) = 0, \tag{4.28}$$

for all $\vec{n}$ such that $n_i = n_{i+1}$ for some $1 \leqslant i \leqslant k$, and satisfies the *free evolution equation*

$$u(t+1,\vec{n}) = \left(\prod_{i=1}^{k} \nabla_i\right) u(t,\vec{n}), \tag{4.29}$$

for all $\vec{n} \in \mathbb{Z}^k$, then the restriction of $u(t,\vec{n})$ to $\mathbb{W}^k$ satisfies the *true evolution equation* (4.26).

Remark 4.3.3. The coefficients $\binom{c}{j}\frac{(\nu-\mu)_j(\mu)_{c-j}}{(\nu)_c}$ that appear in the true evolution equation (4.26) are probabilities of the Beta-binomial distribution with parameters $c, \mu, \nu-\mu$. Hence, the true evolution equation could be interpreted as a the "evolution equation" for a series of urns where each urn evolves according to the Pólya urn scheme. Such dynamics could be interpreted as the $q \to 1$ degeneration of the q-Hahn Boson, which is dual to the q-Hahn TASEP [Cor14].

Proposition 4.3.4. *For $n_1 \geqslant n_2 \geqslant \dots \geqslant n_k \geqslant 1$, one has the following moment formula,*

$$\mathbb{E}\Big[Z(t,n_1)\cdots Z(t,n_k)\Big] = \frac{1}{(2i\pi)^k}\int\cdots\int \prod_{1\leqslant A<B\leqslant k} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} \left(\frac{\nu+z_j}{z_j}\right)^{n_j} \left(\frac{\mu+z_j}{\nu+z_j}\right)^{t} \frac{\mathrm{d}z_j}{\nu+z_j}. \tag{4.30}$$

where the contour for z_k is a small circle around the origin, and the contour for z_j contains the contour for $z_{j+1}+1$ for all $j = 1,\dots,k-1$, as well as the origin, but all contours exclude $-\nu$.

Proof. We show that the right-hand-side of (4.30) satisfies the free evolution equation, the boundary condition and the initial condition for $u(0,\vec{n})$ for $\vec{n} \in \mathbb{W}^k$ (the initial condition outside $\mathbb{W}^k$ is inconsequential). The above discussion shows that the restriction to $\vec{n} \in \mathbb{W}^k$ then solves the true evolution equation (4.26). By the definition of the function u in (4.24) and the initial condition for the half-line to point polymer, $u(0,\vec{n}) = \prod_{i=1}^{k} \mathbb{1}_{n_i \geqslant 1} = \mathbb{1}_{n_k \geqslant 1}$ (the second equality holds because the n_i's are ordered). Let us consider the right-hand-side of (4.30) when $t = 0$. If $n_k \leqslant 0$, there is no pole in zero, so one can shrink the z_k

contour to zero, and consequently $u(\vec{n}) = 0$. When $n_k > 0$ (and consequently all n_i's are positive), there is no pole at ν for $t = 0$, so that one can successively send to infinity the contours for the variables z_k, $z_{k-1}, \dots$. Since the residue at infinity is one for each variable, then $u(\vec{n}) = 1$. Hence, the initial condition is satisfied.

In order to show that the boundary condition is satisfied, we assume that $n_i = n_{i+1}$ for some i. Let us apply the operator

$$\left(\frac{1}{1+\nu}\tau^{(i)}\tau^{(i+1)} + \frac{\nu-1}{1+\nu}\tau^{(i+1)} + \frac{1}{1+\nu} - \tau^{(i)}\right)$$

inside the integrand. This brings into the integrand a factor

$$\frac{1}{1+\nu}\frac{z_i}{\nu+z_i}\frac{z_{i+1}}{\nu+z_{i+1}} + \frac{\nu-1}{\nu+1}\frac{z_{i+1}}{\nu+z_{i+1}} + \frac{1}{1+\nu} - \frac{z_i}{\nu+z_i} = \frac{-\nu^2(z_i - z_{i+1} - 1)}{(1+\nu)(\nu+z_i)(\nu+z_{i+1})}.$$

Since it cancels the pole for $z_i = z_{i+1}+1$, one can use the same contour for both variables, and since the integrand is now antisymmetric in the variables (z_i, z_{i+1}) the integral is zero as desired.

In order to show that the free evolution equation is satisfied, it is enough to show that applying the operator $p\tau^{(i)} + (1-p)$ for i from 1 to k inside the integrand brings an extra factor

$$\prod_{j=1}^{k} \frac{\mu+z_j}{\nu+z_j}.$$

This is clearly true since

$$\left(p\tau^{(i)} + (1-p)\right)\left(\frac{\nu+z_i}{z_i}\right)^{n_i} = \left(\frac{\nu+z_i}{z_i}\right)^{n_i}\frac{\mu+z_i}{\nu+z_i}.$$

□

Remark 4.3.5. It is possible to prove a generalization of Proposition 4.3.4 where the parameter μ depend on t. In this generalization, the weight of an edge starting from a point (s, n) for any n would have a weight B or $1-B$ (depending on the direction of the edge), where B is a random variable distributed according to the Beta distribution with parameters $(\mu_s, \nu - \mu_s)$. In the formula (4.30), the factor $\left(\frac{\mu+z_j}{\nu+z_j}\right)^t$ would be replaced by

$$\prod_{s=0}^{t-1} \frac{\mu_s + z_j}{\nu+z_j}.$$

Such moment formulas with time inhomogeneous parameters have been proved for the discrete-time q-TASEP [BC13] and for the q-Hahn TASEP in [Cor14, Section 2.4] (See also the discussion in [CP15, Section] which deals with a generalization of the q-Hahn TASEP). In all these cases, this allows to prove Fredholm determinant formulas with time-dependent parameters, using the same method as in the homogeneous case. It is not clear however if one can find moment formulas with a parameter inhomogeneity depending on n (e.g. the parameter ν would depend on n).

Proposition 4.3.4 provides an integral formula for the moments of $Z(t, n)$. In order to form the generating series, it is convenient to transform the formula so that all integrations are on the same contour.

Proposition 4.3.6. *For all $n, t \geqslant 0$, we have*

$$\mathbb{E}\left[Z(t,n)^k\right] = k! \sum_{\lambda \vdash k} \frac{1}{m_1! m_2! \dots} \frac{1}{(2i\pi)^{\ell(\lambda)}} \int \dots \int \det\left(\frac{1}{v_j - v_i - \lambda_i}\right)_{i,j=1}^{\ell(\lambda)} \times \prod_{j=1}^{\lambda_j} f(v_j) f(v_j+1) \dots f(v_j + \lambda_j - 1) \mathrm{d}v_1 \dots \mathrm{d}v_{\ell(\lambda)}, \quad (4.31)$$

where

$$f(v) = \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+1)} = \left(\frac{\nu + v}{v}\right)^n \left(\frac{\mu + v}{\nu + v}\right)^t \frac{1}{v + \nu}.$$

where g^{BP} is defined in (4.5) and the integration contour is a small circle around 0 excluding $-\nu$ and for a partition $\lambda \vdash k$ we write $\lambda = 1^{m_1} 2^{m_2} \dots$ (m_i is the number of i components) and $\ell(\lambda)$ is the number of non-zero components $\ell(\lambda) = \sum_i m_i$.

Proof. This type of deduction, called the contour shift argument, has already occurred in the context of the q-Whittaker process in [BC14, Section 3.2.1]. See [BCPS15], in particular Proposition 7.4, and references therein for more background on the contour shift argument. The present formulation corresponds to a degeneration when $q \to 1$ of the Proposition 3.2.1 in [BC14].

One starts with the moment formula given by Proposition 4.3.4:

$$\mathbb{E}\left[Z(t,n)^k\right] = \frac{1}{(2i\pi)^k} \int \dots \int \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} f(z_j) \mathrm{d}z_j.$$

We need to shrink all contours to a small circle around 0. During the deformation of contours, one encounters all poles of the product $\prod_{A<B} \frac{z_A - z_B}{z_A - z_B - 1}$. Thus, a direct proof would amount to carefully book-keeping the residues. Although one could adapt to the present setting the proof of [BCPS15, Proposition 7.4], we refer to Proposition 6.2.7 in [BC14] which provides a very similar formula. The only modification is that the function f that we consider has a pole at $-\nu$, but this does not play any role in the deformation of contours.

It is also worth remarking that applying Proposition 3.2.1 in [BC14] to q-Hahn moment formula [Cor14, Theorem 1.8] and taking a suitable limit yields the statement of Proposition 4.3.6. $\square$

4.3.2 Second proof of Theorem 4.1.12

Thanks to Proposition 4.3.6, the moments of $Z(t,n)$ have a suitable form for taking the generating series. Let us denote $\mu_k = \mathbb{E}\left[Z(t,n)^k\right]$. A degeneration when q goes to 1 of Proposition 3.2.8 in [BC14] shows that

$$\sum_{k \geqslant 0} \mu_k \frac{u^k}{k!} = \det(I + K)_{\mathbb{L}^2(\mathbb{Z}_{>0} \times C_0)},$$

where $\det(I+K)$ is the formal Fredholm determinant expansion of the operator K defined by the integral kernel

$$K(n_1, v_1; n_2, v_2) = \frac{u^{n_1} f(v_1) f(v_1+1) \dots f(v_1 + n_1 - 1)}{v_1 + n_1 - v_2}.$$

Since $f(v+n)$ is uniformly bounded for $v \in C_0$ and $n \geqslant 1$, and $v_1 + n_1 - v_2$ is uniformly bounded away from 0 for $v_1, v_2 \in C_0, n \geqslant 1$, the identity holds also numerically. Since $|Z(t,n)| \leqslant 1$, then one can exchange summation and expectation so that for any $u \in \mathbb{C}$

$$\sum_{k\geqslant 0} \mu_k \frac{u^k}{k!} = \mathbb{E}\left[e^{uZ(t,n)}\right].$$

It is useful to notice that

$$f(v_1)f(v_1+1)\dots f(v_1+n_1-1) = \frac{g^{\mathrm{BP}}(v_1)}{g^{\mathrm{BP}}(v_1+n_1)}.$$

Next, we want to rewrite $\det(I+K)$ as the Fredholm determinant of an operator acting on a single contour. For that purpose we use the following Mellin-Barnes integral formula:

Lemma 4.3.7. *For $u \in \mathbb{C} \setminus \mathbb{R}_{>0}$ with $|u| < 1$,*

$$\sum_{n=1}^{\infty} u^n \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+n)} \frac{1}{v+n-v'} = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(-s)\Gamma(1+s)(-u)^s \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+s)} \frac{\mathrm{d}s}{v+s-v'}, \tag{4.32}$$

where z^s is defined with respect to a branch cut along $z \in \mathbb{R}_{\leqslant 0}$.

Proof. The statement of the Lemma is very similar with [BC14, Lemma 3.2.13].

Since $\operatorname*{Res}_{s=k}(\Gamma(-s)\Gamma(1+s)) = (-1)^{k+1}$, we have that

$$\sum_{n=1}^{\infty} u^n \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+n)} \frac{1}{v+n-v'} = \frac{1}{2i\pi} \int_{\mathcal{H}} \Gamma(-s)\Gamma(1+s)(-u)^s \frac{g^{\mathrm{BP}}(v)}{g^{\mathrm{BP}}(v+s)} \frac{\mathrm{d}s}{v+s-v'}, \tag{4.33}$$

where $\mathcal{H}$ is a negatively oriented integration contour enclosing all positive integers. For the identity to be valid, the L.H.S. of (4.33) must converge, and the contour must be approximated by a sequence of contours $\mathcal{H}_k$ enclosing the integers $1, \dots, k$ such that the integral along the symmetric difference $\mathcal{H} \setminus \mathcal{H}_k$ goes to zero.

The following estimates show that one can chose the contour $\mathcal{H}_k$ as a rectangular contour connecting the points $1/2+i$, $k+1/2+i$, $k+1/2-i$ and $1/2-i$; and the contour $\mathcal{H}$ as the infinite contour from $\infty - i$ to $1/2 - i$ to $1/2 + i$ to $\infty + i$.

We first need an estimate for the Gamma function [EMO+53, Chapter 1, 1.18 (2)]: for any $\delta > 0$

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-1/2}(1 + \mathcal{O}(1/z)) \quad \text{as } |z| \to \infty, \quad |\arg(z)| < \pi - \delta. \tag{4.34}$$

Then recall that

$$g^{\mathrm{BP}}(v+s) = \left(\frac{\Gamma(v+s)}{\Gamma(\nu+v+s)}\right)^n \left(\frac{\Gamma(\nu+v+s)}{\Gamma(\mu+v+s)}\right)^t \Gamma(\nu+v+s).$$

Using (4.34),

$$g^{\mathrm{BP}}(v+s) = \sqrt{2\pi} e^{-\nu-v-s} (\nu+v+s)^{\nu+v+s-1/2} \frac{(\nu+v+s)^{(\nu-\mu)t}}{(\nu+v+s)^{\nu n}} \left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right).$$

It implies that for s going to $\infty e^{i\phi}$ with $\phi \in [-\pi/2, \pi/2]$, $1/g^{\mathrm{BP}}(v+s)$ has exponential decay in $|s|$. Moreover, for s going to $\infty e^{i\phi}$ with $\phi \in [-\pi/2, \pi/2]$ and $\phi \neq 0$,

$$(-u)^s \frac{\pi}{\sin(\pi s)} \frac{1}{v+s-v'}$$

is bounded. Thus, one can freely deform the integration contour $\mathcal{H}$ in (4.33) to become the straight line from $1/2 - i\infty$ to $1/2 + i\infty$. □

This shows that for any $u \in \mathbb{C} \setminus \mathbb{R}_{>0}$ with $|u| < 1$, one has that

$$\mathbb{E}\left[e^{uZ(t,n)}\right] = \det(I + K_u^{\mathrm{BP}})_{\mathbb{L}^2(C_0)}, \tag{4.35}$$

where the kernel K_u^{BP} is defined in the statement of Theorem 4.1.12. One extends the result to any $u \in \mathbb{C}\setminus\mathbb{R}_{>0}$ by analytic continuation. The right-hand-side in (4.35) is analytic since we have already shown in the proof of Proposition 4.2.6 that the Fredholm determinant expansion is absolutely summable and integrable. The left-hand-side is analytic since $|Z(t,n)| < 1$.

4.4 Zero-temperature limit

4.4.1 Proof of Proposition 4.1.10

In this section, we prove that the Bernoulli-Exponential first passage percolation model is the zero-temperature limit of the Beta-RWRE. The zero temperature limit corresponds to sending the parameters α, β of the Beta RWRE to zero.

Proof. We first show how the transition probabilities for the Beta RWRE degenerate in the zero temperature limit.

Lemma 4.4.1. *Fix $a, b > 0$. For $\epsilon > 0$, let B_ϵ be a Beta distributed random variable with parameters $(\epsilon a, \epsilon b)$. We have the convergence in distribution*

$$\Big(-\epsilon\log(B_\epsilon), -\epsilon\log(1-B_\epsilon)\Big) \Longrightarrow \big(\xi E_a, (1-\xi)E_b\big)$$

as ϵ goes to zero, where ξ is a Bernoulli random variable with parameter $b/(a+b)$ and (E_a, E_b) are exponential random variables with parameters a and b, independent of ξ.

Proof. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous bounded functions.

$$\mathbb{E}\Big[f\big(-\epsilon\log(B_\epsilon)\big)g\big(-\epsilon\log(1-B_\epsilon)\big)\Big] = \\ \int_0^1 f\big(-\epsilon\log(x)\big)g\big(-\epsilon\log(1-x)\big)x^{\epsilon a-1}(1-x)^{\epsilon b-1}\frac{\Gamma(\epsilon a+\epsilon b)}{\Gamma(\epsilon a)\Gamma(\epsilon b)}\mathrm{d}x. \tag{4.36}$$

In order to compute the limit of (4.36), we evaluate separately the contribution of the integral between 0 and 1/2, and between 1/2 and 1. By making the change of variable

$z = -\epsilon\log(x)$, we have that

$$\int_0^{1/2} f\big(-\epsilon\log(x)\big)g\big(-\epsilon\log(1-x)\big)x^{\epsilon a-1}(1-x)^{\epsilon b-1}\frac{\Gamma(\epsilon a+\epsilon b)}{\Gamma(\epsilon a)\Gamma(\epsilon b)} = \\ \frac{\Gamma(\epsilon a+\epsilon b)}{\Gamma(\epsilon a)\Gamma(\epsilon b)}\int_{\epsilon\log(2)}^{\infty} f(z)g\big(-\epsilon\log(1-e^{-z/\epsilon})\big)e^{-az}e^{(\epsilon b-1)\log(1-e^{-z/\epsilon})}\mathrm{d}z. \quad (4.37)$$

Since

$$\frac{\Gamma(\epsilon a+\epsilon b)}{\Gamma(\epsilon a)\Gamma(\epsilon b)} \xrightarrow[\epsilon\to 0]{} \frac{ab}{a+b},$$

the limit of the right-hand-side in (4.37) is

$$\frac{b}{a+b}\int_0^{\infty} f(z)g(0)ae^{-az}\mathrm{d}z = \frac{b}{a+b}\mathbb{E}[f(E_a)g(0)].$$

The contribution of the integral in (4.36) between 1/2 and 1 is computed in the same way, and we find that

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{E}\Big[f\big(-\epsilon\log(B_\epsilon)\big)g\big(-\epsilon\log(1-B_\epsilon)\big)\Big] &= \frac{b}{a+b}\mathbb{E}\big[f(E_a)g(0)\big] + \frac{a}{a+b}\mathbb{E}\big[f(0)g(E_b)\big] \\ &= \mathbb{E}\Big[f(\xi E_a)g((1-\xi)E_b)\Big],\end{aligned}$$

which proves the claim. □

Remark 4.4.2. When E_a and E_b are independent or not does not have any importance. However, it is important that E_a and E_b are independent of the Bernoulli random variable ξ.

Let $\alpha_\epsilon = \epsilon a, \beta_\epsilon = \epsilon b$ and $P_\epsilon(t,x)$ the (quenched) distribution function of the endpoint at time t for the Beta random walk with parameters α_ϵ and β_ϵ. Let $T(n,m)$ be the first-passage time in the Bernoulli-Exponential model with parameters a, b.

It is convenient to define the analogue of the set of weights w_e of the Beta polymer in the context of the Beta RWRE. For an edge e in $(\mathbb{Z}_{\geqslant 0})^2$ we define p_e by

$$p_e = \begin{cases} B_{j-i,i+j} & \text{if } e \text{ is the vertical edge } (i,j)\to(i,j+1) \\ 1-B_{j-i,i+j} & \text{if } e \text{ is the horizontal edge } (i,j)\to(i+1,j); \end{cases}$$

where the variables $B_{\cdot,\cdot}$ define the environment of the random walk, and the passage times t_e are defined in (4.3). Lemma 4.4.1 implies that as ϵ goes to zero, we have the weak convergence

$$\min_{\pi:(0,0)\to D_{n,m}}\left\{\sum_{e\in\pi} -\epsilon\log(p_e)\right\} \Rightarrow \min_{\pi:(0,1)\to D_{n,m}}\left\{\sum_{e\in\pi} t_e\right\},$$

where the minimum is taken over up-right paths.

Since the times t_e in the FPP model are either zero or exponential, and there is at most one path with zero passage time, the minimum over paths of $\sum_{e\in\pi} t_e$ is attained for a unique path with probability one. We know by the principle of the largest term that as $\epsilon\to 0$,

$$-\epsilon\log\big(P_\epsilon(n+m,m-n)\big) = -\epsilon\log\left(\sum_{\pi:(0,0)\to D_{n,m}}\exp\left(\sum_{e\in\pi}\log(p_e)\right)\right)$$

has the same limit as

$$\min_{\pi:(0,0)\to D_{n,m}} \left\{ \sum_{e\in\pi} -\epsilon \log(p_e) \right\}.$$

Since the family of rescaled weights $\big(-\epsilon\log(p_e)\big)_e$ weakly converges to $(t_e)_e$, then

$$\min_{\pi:(0,0)\to D_{n,m}} \left\{ \sum_{e\in\pi} -\epsilon \log(p_e) \right\} \Rightarrow \min_{\pi:(0,0)\to D_{n,m}} \left\{ \sum_{e\in\pi} t_e \right\}.$$

Hence for any $n,m \geqslant 0$, $-\epsilon\log(P_\epsilon(t,n)$ weakly converges as ϵ goes to zero to $T(n,m)$. $\square$

4.4.2 Proof of Theorem 4.1.18

Theorem 4.1.18 states that for $r\in\mathbb{R}_{>0}$, one has

$$\mathbb{P}\big(T(n,m)>r\big) = \det(I+K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0')}$$

where C_0' is a small positively oriented circle containing 0 but not $-\nu$, and K_r^{FPP} : $\mathbb{L}^2(C_0')\to\mathbb{L}^2(C_0')$ is defined by its integral kernel

$$K_r^{\mathrm{FPP}}(u,u') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \frac{e^{rs}}{s}\frac{g^{\mathrm{FPP}}(u)}{g^{\mathrm{FPP}}(u+s)}\frac{\mathrm{d}s}{s+u-u'} \tag{4.38}$$

where

$$g^{\mathrm{FPP}}(u) = \left(\frac{a+u}{u}\right)^n \left(\frac{a+u}{a+b+u}\right)^m \frac{1}{u}. \tag{4.39}$$

Proof. The proof splits into two pieces. We first show that under appropriate scalings, the Laplace transform $\mathbb{E}\big[e^{uP_\epsilon(n+m,m-n)}\big]$ converges to $\mathbb{P}\big(T(n,m)\geqslant r\big)$. Then we show that the Fredholm determinant $\det(I+K_u^{\mathrm{BP}})$ from 4.1.13 converges to $\det(I+K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0')}$.

First step: We have an exact formula for $\mathbb{E}\left[e^{uP_\epsilon(n+m,m-n)}\right]$. Let us scale u as $u = -\exp\left(\epsilon^{-1}r\right)$ so that

$$\mathbb{E}\left[e^{uP_\epsilon(n+m,m-n)}\right] = \mathbb{E}\left[\exp\left(-e^{-\epsilon^{-1}(-\epsilon\log(P_\epsilon(n+m,m-n))-r)}\right)\right].$$

If $f_\epsilon(x) := \exp\left(-e^{-\epsilon^{-1}x}\right)$, then the sequence of functions $\{f_\epsilon\}$ maps $\mathbb{R}$ to $(0,1)$, is strictly increasing with a limit of 1 at $+\infty$ and 0 at $-\infty$, and for each $\delta>0$, on $\mathbb{R}\backslash[-\delta,\delta]$ converges uniformly to $\mathbb{1}_{x>0}$. We define the r-shift of f_ϵ as $f_\epsilon^r(x)=f_\epsilon(x-r)$. Then,

$$\mathbb{E}\Big[e^{uP_\epsilon(n+m,m-n)}\Big] = \mathbb{E}\Big[f_\epsilon^r(-\epsilon\log(P_\epsilon(n+m,m-n)))\Big].$$

Since the variable $T(n,m)$ has an atom in zero, we are not exactly in the situation of Lemma 4.1.38 in [BC14], but we can adapt the proof. Let $s<r<u$. By the properties of the functions f_ϵ mentioned above, we have that for any $\eta>0$, there exists an ϵ_0 such that for any $\epsilon<\epsilon_0$,

$$\mathbb{P}\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\geqslant u\Big) \leqslant \mathbb{E}\Big[f_\epsilon^r\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\Big)\Big] \leqslant \mathbb{P}\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\geqslant s\Big).$$

Since we have established the weak convergence of $-\epsilon \log\big(P_\epsilon(n+m, m-n)\big)$ one can take limits as ϵ goes to zero in the probabilities and we find that

$$\begin{aligned}\mathbb{P}\big(T(n,m) \geqslant u\big) &\leqslant \liminf_{\epsilon\to 0} \mathbb{E}\Big[f_\epsilon^r\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\Big)\Big]\\ &\leqslant \limsup_{\epsilon\to 0} \mathbb{E}\Big[f_\epsilon^r\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\Big)\Big] \leqslant \mathbb{P}\big(T(n,m) \geqslant s\big).\end{aligned}$$

Now we take s and u to r and notice that $T(n,m)$ can be decomposed as an atom at zero and an absolutely continuous part. Thus, for any $r > 0$,

$$\mathbb{P}\big(T(n,m) > r\big) = \lim_{\epsilon\to 0} \mathbb{E}\Big[f_\epsilon^r\Big(-\epsilon\log\big(P_\epsilon(n+m,m-n)\big)\Big)\Big].$$

Second step: We shall prove that the limit when ϵ goes to zero of $\mathbb{E}\left[e^{uP_\epsilon(n+m,m-n)}\right]$ is $\det(I + K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0)}$ where K_r^{FPP} is defined as in Theorem 4.1.18. For that we take the limit of the Fredholm determinant K^{RW} from Theorem 4.1.13. Let us use the change of variables

$$v = \epsilon\tilde{v}, \quad v' = \epsilon\tilde{v}', \quad s = \epsilon\tilde{s}.$$

Assuming that the limit of the Fredholm determinant is the Fredholm determinant of the limit, which we prove below, we have to take the limit of $\epsilon K^{RW}(\epsilon\tilde{v}, \epsilon\tilde{v}')$. The factor ϵ in front of K^{RW} is a priori necessary, it comes from the Jacobian of the change of variables $v = \epsilon\tilde{v}$ and $v' = \epsilon\tilde{v}'$. For any $1 > \epsilon > 0$ the kernel $K^{RW}(v, v')$ can be written as an integral over $\frac{1}{2}\epsilon + i\mathbb{R}$ instead of an integral over $\frac{1}{2} + i\mathbb{R}$, since we do not cross any singularity of the integrand during the contour deformation, and the integrand has exponential decay. Thus, one can write

$$\epsilon K^{RW}(\epsilon\tilde{v}, \epsilon\tilde{v}') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\epsilon\pi}{\sin(\pi\epsilon\tilde{s})}(-u)^{\epsilon\tilde{s}} \frac{g^{RW}(\epsilon\tilde{v})}{g^{RW}(\epsilon\tilde{v}+\tilde{s})}\frac{\mathrm{d}\tilde{s}}{\tilde{s}+\tilde{v}-\tilde{v}'}. \tag{4.40}$$

With $u = -\exp\big(\epsilon^{-1}r\big)$, we have that $(-u)^{\epsilon\tilde{s}} = e^{\tilde{s}r}$. Moreover, since

$$\lim_{\epsilon\to 0} \epsilon\Gamma(\epsilon z) = \frac{1}{z},$$

we have that

$$\lim_{\epsilon\to 0} \frac{g^{RW}(\epsilon\tilde{v})}{g^{RW}(\epsilon\tilde{v}+\tilde{s})} = \frac{g^{\mathrm{FPP}}(\tilde{v})}{g^{\mathrm{FPP}}(\tilde{v}+\tilde{s})},$$

where g^{FPP} is defined in (4.39), and

$$\lim_{\epsilon\to 0}\frac{\epsilon\pi}{\sin\epsilon\tilde{s}} = \frac{1}{\tilde{s}}.$$

Because the integrand in (4.38) is not absolutely integrable, one cannot apply dominated convergence directly. Instead, we will split the integral (4.40) into two pieces: the integral over s when $\mathfrak{Im}[\epsilon s] < 1/4$ and the integral over s when $\mathfrak{Im}[\epsilon s] \geqslant 1/4$. Let us begin with some estimates. Since the function $z \mapsto z/\sin(z)$ is holomorphic on a circle of radius $1/2$ around zero, there exists a constant $C > 0$ such that for $s \in 1/2 + i\mathbb{R}$ and $\epsilon > 0$ such that $|\epsilon s| < 1/2$, we have

$$\Big|\frac{\epsilon\pi}{\sin(\pi\epsilon\tilde{s})} - \frac{1}{s}\Big| < C\epsilon.$$

In order to lighten the notations, we denote

$$G(\epsilon,\tilde{s}) = \frac{g^{RW}(\epsilon\tilde{v})}{g^{RW}(\epsilon\tilde{v}+\epsilon\tilde{s})}\frac{1}{\tilde{s}+\tilde{v}-\tilde{v}'}.$$

The variables $\tilde{v}$ and $\tilde{v}'$ are fixed for the moment. We know that $G(\epsilon,\tilde{s})$ is bounded for ϵ close to zero and $\tilde{s}\in 1/2+i\mathbb{R}$. Moreover, there exists a constant $C'>0$ such that for $|\epsilon s|<1/2$,

$$\left| G(\epsilon,\tilde{s}) - \frac{g^{\mathrm{FPP}}(\tilde{v})}{g^{\mathrm{FPP}}(\tilde{v}+\tilde{s})}\frac{1}{\tilde{v}+\tilde{s}-\tilde{v}'}\right| < C'\epsilon.$$

We have the decomposition

$$\frac{1}{2i\pi}\int_{\frac{1}{2}-\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}} \frac{\epsilon\pi}{\sin(\pi\epsilon\tilde{s})}e^{r\tilde{s}}G(\epsilon,\tilde{s})\mathrm{d}\tilde{s} = \frac{1}{2i\pi}\int_{\frac{1}{2}-\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}} \left(\frac{\epsilon\pi}{\sin(\pi\epsilon\tilde{s})}-\frac{1}{\tilde{s}}\right)e^{r\tilde{s}}G(\epsilon,\tilde{s})\mathrm{d}\tilde{s}$$
$$+\frac{1}{2i\pi}\int_{\frac{1}{2}-\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}} \frac{e^{r\tilde{s}}}{\tilde{s}}\Big(G(\epsilon,\tilde{s})-G(0,\tilde{s})\Big)\mathrm{d}\tilde{s} + \frac{1}{2i\pi}\int_{\frac{1}{2}-\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}} \frac{e^{r\tilde{s}}}{\tilde{s}}G(0,\tilde{s})\mathrm{d}\tilde{s}. \quad (4.41)$$

The first integral in the R.H.S of (4.41) can be bounded by

$$C\epsilon\frac{1}{2\pi}\int_{\frac{1}{2}\epsilon-\frac{i}{4}}^{\frac{1}{2}\epsilon+\frac{i}{4}} |\Gamma(1-s)|e^{r/2}|G(\epsilon,s\epsilon^{-1})|\mathrm{d}s,$$

which is $\mathcal{O}(\epsilon)$. The second integral in the R.H.S of (4.41) can be bounded by

$$C'\epsilon\frac{1}{2\pi}\int_{\frac{1}{2}-\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}} \frac{e^{r/2}}{|\tilde{s}|}\mathrm{d}\tilde{s},$$

which is $\mathcal{O}(\epsilon\log(\epsilon^{-1}))$. The third integral in the R.H.S of (4.41) converges to a limit as ϵ goes to zero, even if the integrand is not absolutely integrable. The limit is the improper integral

$$\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty}\frac{e^{r\tilde{s}}}{\tilde{s}}\frac{g^{\mathrm{FPP}}(\tilde{v})}{g^{\mathrm{FPP}}(\tilde{v}+\tilde{s})}\frac{\mathrm{d}\tilde{s}}{\tilde{v}+\tilde{s}-\tilde{v}'} = K_r^{\mathrm{FPP}}(\tilde{v},\tilde{v}').$$

It remains to show that we have made a negligible error when cutting the tails of the integral. We have

$$\frac{1}{2i\pi}\int_{\frac{1}{2}+\frac{i\epsilon^{-1}}{4}}^{\frac{1}{2}+i\infty}\frac{\epsilon\pi}{\sin(\pi\epsilon\tilde{s})}e^{r\tilde{s}}G(\epsilon,\tilde{s})\mathrm{d}\tilde{s} = \frac{1}{2i\pi}\int_{\frac{1}{2}\epsilon+\frac{i}{4}}^{\frac{1}{2}\epsilon+i\infty}\frac{\pi}{\sin(\pi s)}e^{rs\epsilon^{-1}}G(\epsilon,s\epsilon^{-1})\mathrm{d}s =$$
$$\frac{1}{2i\pi}\int_{\frac{1}{2}\epsilon+\frac{i}{4}}^{\frac{1}{2}\epsilon+i\infty}\frac{\pi}{\sin(\pi s)}e^{rs\epsilon^{-1}}\big(G(\epsilon,s\epsilon^{-1})-1\big)\mathrm{d}s + \frac{1}{2i\pi}\int_{\frac{1}{2}\epsilon+\frac{i}{4}}^{\frac{1}{2}\epsilon+i\infty}\frac{\pi}{\sin(\pi s)}e^{rs\epsilon^{-1}}\mathrm{d}s. \quad (4.42)$$

The first integral in the R.H.S of (4.42) goes to zero by dominated convergence, and the second integral in the R.H.S of (4.42) goes to zero by the Riemann-Lebesgue lemma. At this point we have shown that for any $\tilde{v},\tilde{v}'\in C_0$,

$$\lim_{\epsilon\to 0}\epsilon K^{RW}(\epsilon\tilde{v},\epsilon\tilde{v}') = K_r^{\mathrm{FPP}}(\tilde{v},\tilde{v}').$$

Observe now that the kernel $K_r^{\mathrm{FPP}}(\tilde{v}, \tilde{v}')$ is bounded as $\tilde{v}, \tilde{v}'$ vary along their contour. Using Hadamard's bound, one can bound the Fredholm series expansion of K_r^{FPP} by an absolutely convergent series of integrals, and conclude by dominated convergence that under the scalings above

$$\det(I + K_u^{\mathrm{RW}})_{\mathbb{L}^2(C_0)} \xrightarrow[\epsilon\to 0]{} \det(I + K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0)}.$$

□

4.5 Asymptotic analysis of the Beta RWRE

Let us first define the Tracy-Widom distribution governing the fluctuations of extreme eigenvalues of Gaussian hermitian random matrices. We refer to [BC14, Section 3.2.2] for an introduction to Fredholm determinants.

Definition 4.5.1. The distribution function $F_{\mathrm{GUE}}(x)$ of the GUE Tracy-Widom distribution is defined by $F_{\mathrm{GUE}}(x) = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)}$ where K_{Ai} is the Airy kernel,

$$K_{\mathrm{Ai}}(u,v) = \frac{1}{(2i\pi)^2}\int_{e^{-2i\pi/3}\infty}^{e^{2i\pi/3}\infty} \mathrm{d}w \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \mathrm{d}z \frac{e^{z^3/3 - zu}}{e^{w^3/3-wv}} \frac{1}{z-w},$$

where the contours for z and w do not intersect. There is some freedom in the choice of contours. For instance, one can choose the contour for z (resp. w) as constituted of two infinite rays departing 1 (resp. 0) in directions $\pi/3$ and $-\pi/3$ (resp. $2\pi/3$ and $-2\pi/3$).

4.5.1 Fredholm determinant asymptotics

We consider a Beta RWRE $(X_t)_{t\geqslant 0}$ with parameters $\alpha, \beta > 0$. For a parameter $\theta > 0$, we define the quantity

$$x(\theta) = \frac{\Psi_1(\theta+\alpha+\beta) + \Psi_1(\theta) - 2\Psi_1(\theta+\alpha)}{\Psi_1(\theta) - \Psi_1(\theta+\alpha+\beta)} \tag{4.43}$$

and the function $I : \left(\frac{\alpha-\beta}{\alpha+\beta}, 1\right) \to \mathbb{R}_{>0}$ such that

$$I\big(x(\theta)\big) = \frac{\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta+\alpha)}{\Psi_1(\theta) - \Psi_1(\theta+\alpha+\beta)}\Big(\Psi(\theta+\alpha+\beta) - \Psi(\theta)\Big) + \Psi(\theta+\alpha+\beta) - \Psi(\theta+\alpha), \tag{4.44}$$

where Ψ is the digamma function ($\Psi(z) = \Gamma'(z)/\Gamma(z)$) and Ψ_1 is the trigamma function ($\Psi_1(z) = \Psi'(z)$). Moreover, we define a real-valued $\sigma(\theta) > 0$ such that

$$2\sigma(\theta)^3 = \Psi_2(\theta+\alpha) - \Psi_2(\alpha+\beta+\theta) + \frac{\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)}{\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)}\left(\Psi_2(\alpha+\beta+\theta) - \Psi_2(\theta)\right). \tag{4.45}$$

The fact that we can choose $\sigma(\theta) > 0$ is proved in Lemma 4.5.3. We will see that a critical point Fredholm determinant asymptotic analysis shows that for all $\theta > 0$ and $\alpha, \beta > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\log\Big(P\big(t, x(\theta)t\big)\Big) + I\big(x(\theta)\big)t}{t^{1/3}\sigma(\theta)} \leqslant y\right) = F_{\mathrm{GUE}}(y). \tag{4.46}$$

However, due to increased technical challenges in the general parameter case, we presently prove rigorously only the case of Theorem 4.5.2, which deals with $\alpha = \beta = 1$ (i.e. when the $B_{x,t}$ variables are distributed uniformly on $(0,1)$).

When $\alpha = \beta$ the expressions for $x(\theta)$ and $I(x(\theta))$ simplify. We find that

$$x(\theta) = \frac{1+2\theta}{\theta^2 + (\theta+1)^2}$$

and

$$I\big(x(\theta)\big) = \frac{1}{\theta^2 + (\theta+1)^2},$$

so that the rate function I is simply the function $I : x \mapsto 1 - \sqrt{1-x^2}$. We also find that for $\alpha = \beta = 1$,

$$\sigma(\theta)^3 = \frac{1}{\theta + 3\theta^2 + 4\theta^3 + 2\theta^4} = \frac{2\left(1-\sqrt{1-x^2}\right)^2}{\sqrt{1-x^2}} = \frac{2I(x)^2}{1-I(x)}, \tag{4.47}$$

where $x = x(\theta)$.

Theorem 4.5.2. *For $0 < \theta < 1/2$ and $\alpha = \beta = 1$, we have that*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\log\Big(P\big(t, x(\theta)t\big)\Big) + I\big(x(\theta)\big)t}{t^{1/3}\sigma(\theta)} \leqslant y \right) = F_{\mathrm{GUE}}(y). \tag{4.48}$$

The rest of this section is devoted to the proof of Theorem 4.5.2. Most arguments in the proof apply equally for any parameters α, β except the deformation of contours which is valid for small θ and Lemma 4.5.5 which is only valid for $\alpha = \beta = 1$. We expect the general α, β, θ to still hold but do not attempt to extend to that case.

We first observe that we do not need to invert the Laplace transform of $P(t, x(\theta)t)$. Setting $u = -e^{tI(x(\theta)) - t^{1/3}\sigma(\theta)y}$, one has that

$$\lim_{t\to\infty} \mathbb{E}\left[e^{uP(t,x)}\right] = \lim_{t\to\infty} \mathbb{P}\left(\frac{\log\Big(P\big(t, x(\theta)t\big)\Big) + I\big(x(\theta)\big)t}{t^{1/3}\sigma(\theta)y} < y \right). \tag{4.49}$$

This convergence is justified by Lemma 4.1.39 in [BC14], provided that the limit is a continuous probability distribution function, and we see later that this is the case. Hence, in order to prove Theorem 4.5.2, one has to take the $t \to \infty$ limit of the Fredholm determinant (4.6) in the statement of Theorem 4.1.13.

The asymptotic analysis of this Fredholm determinant proceeds by steepest descent analysis, and is very close to the analysis presented in the recent papers [BCF14, FV13, Bar15, Vet14, CSS15, BC15a], that deal with similar kernels. Let us assume for the moment that the contour C_0 is a circle around 0 with very small radius. One can make the change of variables $v + s = z$ in the kernel K_u^{RW} so that, with the value of u that we choose,

$$K_u^{\mathrm{RW}}(v, v') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi(z-w))} e^{(z-w)(tI(x(\theta)) - t^{1/3}\sigma(\theta)y)} \frac{g^{\mathrm{RW}}(v)}{g^{\mathrm{RW}}(z)} \frac{\mathrm{d}z}{z - v'},$$

and the contour for z can be chosen as $1/2+i\mathbb{R}$. The kernel can be rewritten

$$K_u^{\mathrm{RW}}(v,v') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi(z-w))} \exp\Big(t(h(z)-h(v)) - t^{1/3}\sigma(\theta)y(z-v)\Big) \frac{\Gamma(v)}{\Gamma(z)} \frac{\mathrm{d}z}{z-v'}, \tag{4.50}$$

where

$$h(z) = I\big(x(\theta)\big)z + \frac{1-x(\theta)}{2}\log\left(\frac{\Gamma(\alpha+z)}{\Gamma(z)}\right) + \frac{1+x(\theta)}{2}\log\left(\frac{\Gamma(\alpha+z)}{\Gamma(\alpha+\beta+z)}\right).$$

The function h governs the asymptotic behaviour of the Fredholm determinant of K_u^{RW}. The principle of the steepest-descent method is to deform the integration contour – both the contour in the definition of K_u^{RW} and the $\mathbb{L}^2$ contour – so that they go across a critical point of the function h. Then one needs to prove that only the integration around the critical point has a contribution in the limit, and one can approximate all terms by their Taylor approximation close to the critical point.

The first derivatives of h are

$$h'(z) = I\big(x(\theta)\big) + \Psi(\alpha+z) - \frac{1}{2}\Psi(z) - \frac{1}{2}\Psi(\alpha+\beta+z) + \frac{x(\theta)}{2}\Big(\Psi(z) - \Psi(\alpha+\beta+z)\Big),$$

and

$$h''(z) = \Psi_1(\alpha+z) - \frac{1}{2}\Psi_1(z) - \frac{1}{2}\Psi_1(\alpha+\beta+z) + \frac{x(\theta)}{2}\Big(\Psi_1(z) - \Psi_1(\alpha+\beta+z)\Big).$$

One readily sees that the expressions for $x(\theta)$ and $I(x(\theta))$ in (4.43) and (4.44) are precisely chosen so that $h'(\theta) = h''(\theta) = 0$. Let us give an expression of h' in terms of θ:

$$\begin{aligned} h'(z) &= \Psi(z+\alpha) - \Psi(\alpha+\beta+z) + \frac{\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)}{\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)}\Big(\Psi(\alpha+\beta+z) - \Psi(z)\Big) \\ &- \left(\Psi(\theta+\alpha) - \Psi(\alpha+\beta+\theta) + \frac{\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)}{\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)}\Big(\Psi(\alpha+\beta+\theta) - \Psi(\theta)\Big)\right). \end{aligned} \tag{4.51}$$

Expressions are much simpler in the case $\alpha = \beta = 1$. In that case we have

$$\begin{aligned} h'(z) &= \frac{1}{\theta+1} - \frac{1}{z+1} + \frac{1}{1+\left(\frac{\theta+1}{\theta}\right)^2}\left(\frac{2z+1}{z(z+1)} - \frac{2\theta+1}{\theta(\theta+1)}\right), \\ &= \frac{(\theta-z)^2}{z(1+z)(1+2\theta+2\theta^2)}. \end{aligned} \tag{4.52}$$

In order to understand the behaviour of $\mathfrak{Re}[h]$ around the critical point θ, we also need the sign of the third derivative of h.

Lemma 4.5.3. *For any $\alpha, \beta, \theta > 0$, we have that $h'''(\theta) > 0$.*

Lemma 4.5.3 is proved in Section 4.5.2.

By the definition of $\sigma(\theta)$ in (4.45), $\sigma(\theta) = \left(\frac{h'''(\theta)}{2}\right)^{1/3}$. Then, using Taylor expansion, we have that for z in a neighbourhood of θ,

$$h(z) - h(\theta) \approx \frac{\big(\sigma(\theta)(z-\theta)\big)^3}{3}. \tag{4.53}$$

We now deform the integration contour in (4.50) and the Fredholm determinant contour which was initially a small circle around 0. Let $\mathcal{D}_\theta$ be the vertical line $\mathcal{D}_\theta = \{\theta + iy : y \in \mathbb{R}\}$, and $\mathcal{C}_\theta$ be the circle centred in 0 with radius θ. This deformation of contours does not change the Fredholm determinant $\det(I + K_u^{\mathrm{RW}})$ only if

- All the poles of the sine inverse in (4.50) corresponding with $z - w \in \mathbb{Z}_{\geqslant 1}$ stay on the right of $\mathcal{D}_\theta$.
- We do not cross the pole of h at $-\alpha - \beta$ when deforming the $\mathbb{L}^2$ contour.

Hence, we will assume that $\theta < \min(\alpha+\beta, \frac{1}{2})$ so that the two above conditions are satisfied.

Lemma 4.5.4. *For any parameters $\alpha, \beta > 0$, and $\theta > 0$, the contour $\mathcal{D}_\theta$ is steep-descent for the function $\mathfrak{Re}[h]$ in the sense that $y \mapsto \mathfrak{Re}[h(\theta + iy)]$ is decreasing for y positive and increasing for y negative.*

Lemma 4.5.4 is proved in Section 4.5.2. The step which prevents us from proving Theorem 4.5.2 for any parameters $\alpha, \beta > 0$ is the steep-descent properties of the contour $\mathcal{C}_\theta$.

Lemma 4.5.5. *Assume $\alpha = \beta = 1$. Then the contour $\mathcal{C}_\theta$ is steep descent for the function $-\mathfrak{Re}[h]$, in the sense that $y \mapsto \mathfrak{Re}[h(\theta e^{i\phi})]$ is increasing for $\phi \in (0, \pi)$ and decreasing for $\phi \in (-\pi, 0)$.*

Lemma 4.5.5 is proved in Section 4.5.2. Proving Lemma 4.5.5 for arbitrary parameters α, β turns out to be computationally difficult, and we do not pursue that here.

In the rest of this section, although the proofs are quite general and do not depend on the value of parameters, we assume that $\alpha = \beta = 1$ so that we can use Lemma 4.5.5. Let us show that the only part of the contours that contributes to the limit of the Fredholm determinant when t tends to infinity is a neighbourhood of the critical point θ.

Proposition 4.5.6. *Let $B(\theta, \epsilon)$ be the ball of radius ϵ centred at θ. We note $\mathcal{C}_\theta^\epsilon$ (resp. $\mathcal{D}_\theta^\epsilon$) the part of the contour $\mathcal{C}_\theta$ (resp. $\mathcal{D}_\theta$) inside the ball $B(\theta, \epsilon)$. Then, for any $\epsilon > 0$,*

$$\lim_{t\to\infty} \det\left(I + K_u^{\mathrm{RW}}\right)_{\mathbb{L}^2(\mathcal{C}_\theta)} = \lim_{t\to\infty} \det\left(I + K_{y,\epsilon}^{\mathrm{RW}}\right)_{\mathbb{L}^2(\mathcal{C}_\theta^\epsilon)}$$

where $K_{y,\epsilon}^{\mathrm{RW}}$ is defined by the integral kernel

$$K_{y,\epsilon}^{\mathrm{RW}}(v, v') = \frac{1}{2i\pi} \int_{\mathcal{D}_\theta^\epsilon} \frac{\pi}{\sin(\pi(z-w))} \exp\Big(t(h(z) - h(v)) - t^{1/3}\sigma(\theta) y (z - v)\Big) \frac{\Gamma(v)}{\Gamma(z)} \frac{\mathrm{d}z}{z - v'}. \tag{4.54}$$

Proof. By Lemmas 4.5.4 and 4.5.5, there exists a constant $C > 0$ such that if $v \in \mathcal{C}_\theta$ and $z \in \mathcal{D}_\theta \setminus \mathcal{D}_\theta^\epsilon$, then

$$\mathrm{Re}[h(z) - h(v)] < -C.$$

and consequently

$$\exp\Big(t(h(z)-h(v)) - t^{1/3}\sigma(\theta)y(z-v)\Big)\frac{\mathrm{d}z}{z-v'} \xrightarrow[t\to\infty]{} 0.$$

Since $\frac{\pi}{\sin(\pi(z-w))\Gamma(z)}$ has exponential decay in the imaginary part of z, the contribution of the integration over $\mathcal{D}_\theta \setminus \mathcal{D}_\theta^\epsilon$ is negligible (by dominated convergence). Thus, $K_y^{\mathrm{RW}}(v,v')$ and $K_{y,\epsilon}^{\mathrm{RW}}(v,v')$ have the same limit when t goes to infinity.

By Lemmas 4.5.4 and 4.5.5, there exists another constant $C'>0$ such that if $v\in\mathcal{C}_\theta\backslash\mathcal{C}_\theta^\epsilon$ and $z\in\mathcal{D}_\theta$, then

$$\mathrm{Re}[h(z)-h(v)] < -C'.$$

Consider the Fredholm determinant expansion

$$\det\left(I+K_u^{\mathrm{RW}}\right) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int\dots\int\det\Big(K_u^{\mathrm{RW}}(w_i,w_j)\Big)_{i,j=1}^n \mathrm{d}w_1\dots\mathrm{d}w_n.$$

The k^{th} term can be decomposed as the sum of the integration over $(\mathcal{C}_\theta^\epsilon)^k$ plus the integration over $(\mathcal{C}_\theta)^k\setminus(\mathcal{C}_\theta^\epsilon)^k$. The second contribution goes to zero since it will be possible to factorize $e^{-C't}$. Finally, the proposition is proved using again dominated convergence on the Fredholm series expansion, which is absolutely summable by Hadamard's bound. □

Let us rescale the variables around θ by the change of variables

$$z=\theta+t^{-1/3}\tilde{z},\quad v=\theta+t^{-1/3}\tilde{v},\quad v'=\theta+t^{-1/3}\tilde{v}'.$$

The Fredholm determinant of $K_{y,\epsilon}^{\mathrm{RW}}$ on the contour $\mathcal{C}_\theta^\epsilon$ equals the Fredholm determinant of the rescaled kernel

$$K_{y,\epsilon}^t(\tilde{v},\tilde{v'}) = t^{-1/3}K_{y,\epsilon}^{\mathrm{RW}}\left(\theta+t^{-1/3}\tilde{v},\theta+t^{-1/3}\tilde{v}'\right)$$

acting on the contour $\mathcal{C}_\theta^{t^{1/3}\epsilon}$.

It is more convenient to change again the contours. For $L\in\mathbb{R}_{>0}$, define the contour

$$\mathcal{C}^L := \left\{|y|e^{i(\pi-\phi)\cdot\mathrm{sgn}(y)} : y\in[0,L]\right\}, \tag{4.55}$$

where ϕ is some angle $\phi\in(\pi/6,\pi/2)$ to be chosen later. We also set

$$\mathcal{C} := \left\{|y|e^{i(\pi-\phi)\cdot\mathrm{sgn}(y)} : y\geqslant 0\right\}. \tag{4.56}$$

The contour $\mathcal{C}_\theta^\epsilon$ is an arc of circle and crosses θ vertically. For ϵ small enough, one can replace the contour $\mathcal{C}_\theta^\epsilon$ by $\mathcal{C}^L$ without changing the Fredholm determinants. The values of L and ϕ has to be chosen so that the endpoints of the contours coincide.

We define the rescaled contour for the variable $\tilde{z}$ by

$$\mathcal{D}^L := \{iy : y\in[-L,L]\},$$

and we set $\mathcal{D} := i\mathbb{R}$.

Proposition 4.5.7. *We have that*

$$\lim_{t\to\infty} \det(I + K^{\mathrm{BP}}_{y,\epsilon})_{\mathbb{L}^2(\mathcal{C}^\epsilon_\theta)} = \det(I - K_y)_{\mathbb{L}^2(\mathcal{C})},$$

where K_y is defined by its integral kernel

$$K_y(w, w') = \frac{1}{2i\pi} \int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} \frac{\mathrm{d}z}{(z - w')(w - z)} \frac{e^{z^3/3 - yz}}{e^{w^3/3 - yw}}$$

where the contour for z is a wedge-shaped contour constituted of two rays going to infinity in the directions $e^{-i\pi/3}$ and $e^{i\pi/3}$, such that it does not intersect $\mathcal{C}$.

The proof of Proposition 4.5.7 follows the lines of [FV13, Proposition 6.4] (see also [BC15a, Proposition 6.13]).

Proof. We take the limit of the rescaled kernel $\det(I + K^t_{y,\epsilon}(\tilde{v}, \tilde{v}'))$. Let us first examine the pointwise convergence. Under the scalings above

$$\begin{aligned}
\frac{t^{-1/3}\pi}{\sin(\pi(z - v))} &\xrightarrow[t\to\infty]{} \frac{1}{\tilde{z} - \tilde{v}},\\
\frac{\mathrm{d}z}{z - v'} &\xrightarrow[t\to\infty]{} \frac{\mathrm{d}\tilde{z}}{\tilde{z} - \tilde{v}'},\\
\frac{\Gamma(v)}{\Gamma(z)} &\xrightarrow[t\to\infty]{} 1,\\
t(h(z) - h(v)) &\xrightarrow[t\to\infty]{} \frac{\sigma(\theta)^3}{3}(\tilde{z}^3 - \tilde{v}^3).
\end{aligned}$$

Now we justify that one can take the pointwise limit. We take $\mathcal{D}^{\epsilon t^{1/3}}$ as the integration contour for the $\tilde{z}$ variable. Since $\tilde{z}$ is pure imaginary, $\exp(\tilde{z}^3/3 - \tilde{z}y\sigma(\theta))$ has modulus one. Moreover for fixed $\tilde{v}$ and $\tilde{v}'$, we can find a constant $C''' > 0$ such that

$$\frac{t^{-1/3}\pi}{\sin(\pi(z - v))} \frac{\mathrm{d}z}{z - v'} < \frac{C'''}{(\mathfrak{Im}(\tilde{z})^2)}.$$

This means that the integrand of $K^t_{y,\epsilon}(\tilde{v}, \tilde{v'})$ has quadratic decay, which is enough to apply dominated convergence. It results that

$$\lim_{t\to\infty} K^t_{y,\epsilon}(\tilde{v}, \tilde{v'}) = \frac{1}{2i\pi} \int_{\mathcal{D}^\infty} \frac{e^{\tilde{z}^3\sigma(\theta)^3/3 - \tilde{z}y\sigma(\theta)}}{e^{\tilde{v}^3\sigma(\theta)^3/3 - \tilde{v}y\sigma(\theta)}} \frac{1}{\tilde{z} - \tilde{v}} \frac{\mathrm{d}\tilde{z}}{\tilde{z} - \tilde{v}'}.$$

Now we need to prove that one can exchange the limit with the Fredholm determinant. By Taylor expansion, there exists a constant $C > 0$ such that for $|v - \theta| < \epsilon$,

$$\left| t \cdot h(v) - \frac{\sigma(\theta)^3}{3}(\tilde{v})^3 \right| < Ct(v - \theta)^4. \tag{4.57}$$

Since $|v - \theta| < \epsilon$, we have that $Ct(v - \theta)^4 < C\epsilon\tilde{v}^3$. Hence, for ϵ small enough, one can factor out $\exp(-C'\tilde{v}^3/3)$ for some $C' > 0$. By using the same bound as before for the factors in the integrand of $K^t_{y,\epsilon}$, there exist constants $C', C'' > 0$ such that

$$K^t_{y,\epsilon}(\tilde{v}, \tilde{v'}) < C'' \exp(C'\tilde{v}^3).$$

As $\exp(-\tilde{v}^3)$ decays exponentially in the direction $\infty e^{\pm i\phi}$ for $\phi \in (\pi/2, 5\pi/6)$, we have that for ϵ small enough, the integrand of the rescaled kernel decays exponentially and we can apply dominated convergence. Now recall that we can take ϵ arbitrarily small in Proposition 4.5.6. Thus, the Fredholm expansion of K^t is integrable and summable (using Hadamard's bound), and dominated convergence implies that the limit of $\det(I + K^{\mathrm{BP}}_{y,\epsilon})_{\mathbb{L}^2(\mathcal{C}^\epsilon_\theta)}$ is the Fredholm determinant of an operator $\tilde{K}_y$ acting on $\mathcal{C}$ defined by the integral kernel

$$\tilde{K}_y(\tilde{v}, \tilde{v}') = \frac{1}{2i\pi}\int_{\mathcal{D}^\infty} \frac{e^{\tilde{z}^3\sigma(\theta)^3/3 - \tilde{z}y\sigma(\theta)}}{e^{\tilde{v}^3\sigma(\theta)^3/3 - \tilde{v}y\sigma(\theta)}} \frac{1}{\tilde{z}-\tilde{v}} \frac{\mathrm{d}\tilde{z}}{\tilde{z}-\tilde{v}'}.$$

Since the integrand of $\tilde{K}_y$ has quadratic decay on the tails of the contour $\mathcal{D}^\infty$ one can freely deform the contours so that it goes from $\infty e^{-i\pi/3}$ to $\infty e^{i\pi/3}$ without intersecting $\mathcal{C}^\infty$. Finally, by doing another change of variables to eliminate the dependency in $\sigma(\theta)$ in the integrand, one recovers the Fredholm determinant of K_y as claimed. □

Using the $\det(I + AB) = \det(I + BA)$ trick, one can reformulate the Fredholm determinant of K_y as the Fredholm determinant of a operator on $\mathbb{L}^2(y, \infty)$ (see e.g. [BCF14, Lemma 8.6]). It turns out that

$$\det(I - K_y)_{\mathbb{L}^2(\mathcal{C})} = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x, +\infty)},$$

and this concludes the proof of Theorem 4.5.2.

4.5.2 Precise estimates and steep-descent properties

The following series representations will be useful:

$$\Psi(z) - \Psi(w) = \sum_{n=0}^{\infty} \frac{z-w}{(n+z)(n+w)}, \tag{4.58}$$

is valid for z and w away from the negative integers. We also use

$$\Psi_1(z) - \Psi_1(w) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} - \frac{1}{(n+w)^2}. \tag{4.59}$$

Proof of Lemma 4.5.3. Given the expression (4.51) for the first derivative of h, we have

$$h'''(\theta) = \Psi_2(\theta+\alpha) - \Psi_2(\alpha+\beta+\theta) + \frac{\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)}{\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)}\Big(\Psi_2(\alpha+\beta+\theta) - \Psi_2(\theta)\Big), \tag{4.60}$$

where Ψ_2 is the second polygamma function ($\Psi_2(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Psi_1(z)$). Hence $h'''(\theta) > 0$ is equivalent to

$$\Big(\Psi_2(\theta+\alpha+\beta) - \Psi_2(\theta+\alpha)\Big)\Big(\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta)\Big) - \Big(\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta+\alpha)\Big)\Big(\Psi_2(\theta+\alpha+\beta) - \Psi_2(\theta)\Big) > 0,$$

which is equivalent to

$$\frac{\Psi_2(\theta+\alpha+\beta) - \Psi_2(\theta+\alpha)}{\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta+\alpha)} > \frac{\Psi_2(\theta+\alpha+\beta) - \Psi_2(\theta)}{\Psi_1(\theta+\alpha+\beta) - \Psi_1(\theta)}. \tag{4.61}$$

The function trigamma Ψ_1 is positive and decreasing on $\mathbb{R}_{>0}$. The function Ψ_2 is negative and increasing. One recognizes in (4.61) difference quotients for the function $\Psi_2 \circ \Psi_1^{-1}$. Thus, it is enough to prove that $\Psi_2 \circ \Psi_1^{-1}$ is strictly concave. The derivative of $\Psi_2 \circ \Psi_1^{-1}$ is $\Psi_3 \circ \Psi_1^{-1} / \Psi_2 \circ \Psi_1^{-1}$. Since Ψ_1 is decreasing, it is enough to show that Ψ_3/Ψ_2 is increasing, which, by taking the derivative, is equivalent to $\Psi_4 \Psi_2 > \Psi_3 \Psi_3$.

For all $n \geqslant 1$, one has the integral representation

$$\Psi_n(x) = -\int_0^\infty \frac{(-t)^n e^{-xt}}{1-e^{-t}} \mathrm{d}t. \tag{4.62}$$

Thus for $x > 0$, $\Psi_4(x)\Psi_2(x) > \Psi_3(x)\Psi_3(x)$ is equivalent to

$$\int_0^\infty \int_0^\infty \frac{e^{-xt-xu}}{(1-e^{-t})(1-e^{-u})} t^3 u^3 < \int_0^\infty \int_0^\infty \frac{e^{-xt-xu}}{(1-e^{-t})(1-e^{-u})} t^2 u^4.$$

By symmetrizing the right-hand-side, the inequality is equivalent to

$$\int_0^\infty \int_0^\infty \frac{e^{-xt-xu} t^2 u^2}{(1-e^{-t})(1-e^{-u})} tu < \int_0^\infty \int_0^\infty \frac{e^{-xt-xu} t^2 u^2}{(1-e^{-t})(1-e^{-u})} \frac{t^2+u^2}{2},$$

which is true for all $x > 0$. ☐

Proof of Lemma 4.5.4. By symmetry, it is enough to treat only the case $y > 0$. Hence we show that if $y > 0$, then $\mathfrak{Im}[h'(\theta + iy)] > 0$. Using (4.51), $\mathfrak{Im}[h'(\theta + iy)] > 0$ is equivalent to

$$\begin{aligned} &\Big(\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)\Big)\mathfrak{Im}\Big[\Psi(\alpha+\theta+iy) - \Psi(\alpha+\beta+\theta+iy)\Big] \\ &\quad + \Big(\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)\Big)\mathfrak{Im}\Big[\Psi(\alpha+\beta+\theta+iy) - \Psi(\theta+iy)\Big] > 0. \end{aligned} \tag{4.63}$$

Using the series representations (4.58), Equation (4.63) is equivalent to

$$\begin{aligned} &\Big(\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)\Big)\mathfrak{Im}\sum_{m=0}^{\infty} \frac{-\beta}{(m+\theta+\alpha+iy)(m+\theta+\alpha+\beta+iy)} \\ &+ \Big(\Psi_1(\alpha+\theta) - \Psi_1(\alpha+\beta+\theta)\Big)\mathfrak{Im}\sum_{m=0}^{\infty} \frac{\alpha+\beta}{(m+\theta+iy)(m+\theta+\alpha+\beta+iy)} > 0, \end{aligned} \tag{4.64}$$

We have that

$$\mathfrak{Im}\left[\frac{-\beta}{(m+\theta+\alpha+iy)(m+\theta+\alpha+\beta+iy)}\right] = \frac{1}{(m+\theta+\alpha)^2+y^2} - \frac{1}{(m+\theta+\alpha+\beta)^2+y^2}$$

and

$$\mathfrak{Im}\left[\frac{-(\alpha+\beta)}{(m+\theta+iy)(m+\theta+\alpha+\beta+iy)}\right] = \frac{1}{(m+\theta)^2+y^2} - \frac{1}{(m+\theta+\alpha+\beta)^2+y^2}.$$

It yields that (4.64) can be rewritten as

$$\begin{aligned} &\Big(\Psi_1(\theta) - \Psi_1(\alpha+\beta+\theta)\Big)\Big(\Phi(\theta+\alpha) - \Phi(\theta+\alpha+\beta)\Big) > \\ &\qquad \Big(\Psi_1(\theta) + \alpha - \Psi_1(\alpha+\beta+\theta)\Big)\Big(\Phi(\theta) - \Phi(\theta+\alpha+\beta)\Big), \end{aligned} \tag{4.65}$$

where

$$\Phi(x) = \sum_{n\geqslant 0} \frac{1}{(n+x)^2+y^2}.$$

The inequality (4.65) is equivalent to

$$\frac{\Psi_1(\theta)-\Psi_1(\theta+\alpha)}{\Phi(\theta)-\Phi(\theta+\alpha)} > \frac{\Psi_1(\theta+\alpha)-\Psi_1(\theta+\alpha+\beta)}{\Phi(\theta+\alpha)-\Phi(\theta+\alpha+\beta)} \tag{4.66}$$

Using Cauchy's mean value theorem, there exist $\theta_1 \in (\theta, \theta+\alpha)$ and $\theta_2 \in (\theta+\alpha, \theta+\alpha+\beta)$ such that (4.66) is equivalent to

$$\frac{\Psi_2(\theta_1)}{\Phi'(\theta_1)} > \frac{\Psi_2(\theta_2)}{\Phi'(\theta_2)}.$$

Finally, this last inequality is always true for $\theta_1 < \theta_2$ since we have the series of equivalences

$$\begin{aligned}
&\Psi_2(\theta_1)\Phi'(\theta_2) > \Psi_2(\theta_1)\Phi'(\theta_2) \\
\Leftrightarrow\ & \sum_{n=0}^{\infty} \frac{2}{(n+\theta_1)^3} \sum_{m=0}^{\infty} \frac{2(m+\theta_2)}{\left((m+\theta_2)^2+y^2\right)^2} > \sum_{n=0}^{\infty} \frac{2}{(n+\theta_2)^3} \sum_{m=0}^{\infty} \frac{2(m+\theta_1)}{\left((m+\theta_1)^2+y^2\right)^2} \\
\Leftrightarrow\ & \sum_{n,m=0}^{\infty} \frac{1}{(n+\theta_1)^3(n+\theta_2)^3} \frac{1}{1+\frac{2y^2}{(m+\theta_2)^2}+\frac{y^2}{(m+\theta_2)^4}} > \\
&\qquad \sum_{n,m=0}^{\infty} \frac{1}{(n+\theta_1)^3(n+\theta_2)^3} \frac{1}{1+\frac{2y^2}{(m+\theta_1)^2}+\frac{y^2}{(m+\theta_1)^4}}.
\end{aligned} \tag{4.67}$$

The inequality (4.67) is satisfied because $\theta_1 < \theta_2$. □

Proof of Lemma 4.5.5 in the case $\alpha=\beta=1$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathfrak{Re}\left[h(\theta e^{i\phi})\right] = \mathfrak{Re}\left[i\theta e^{i\phi} h'(\theta e^{i\phi})\right].$$

Using formula (4.52), we have

$$h'(\theta e^{i\phi}) = \frac{\theta(1-e^{i\phi})^2}{e^{i\phi}(\theta e^{i\phi}+1)\left((\theta+1)^2+\theta^2\right)}.$$

We have to show that for any $\phi \in (0,\pi)$, $\mathfrak{Re}\left[i\theta e^{i\phi}h'(\theta e^{i\phi})\right] > 0$. We can forget the factor $\theta/\left((\theta+1)^2+\theta^2)\right)$ which is positive. Thus, we have to show that

$$\mathfrak{Im}\left[\frac{(1-e^{i\phi})^2}{(\theta e^{i\phi}+1)}\right] < 0.$$

One can see that the inequality is equivalent to

$$2\sin(\phi)(\cos(\phi)-1) < 0,$$

which is always true for $\phi \in (0,\pi)$. □

4.5.3 Relation to extreme value theory

Let us now state a corollary of Theorem 4.5.2. Let $(X_t^{(1)})_{t\in\mathbb{Z}_{\geqslant 0}}, \dots, (X_t^{(N)})_{t\in\mathbb{Z}_{\geqslant 0}}$ be N independent random walks drawn in the same Beta environment (Definition 4.1.1). We denote by $\mathcal{P}$ and $\mathcal{E}$ the measure and expectation associated with the probability space which is the product of the environment probability space and the N random walks probability space (for f a function of the environment and the N random walk paths, we have $\mathcal{E}[f] = \mathbb{E}\left[\mathsf{E}^{\otimes N}[f]\right]$ and $\mathcal{P}(A) = \mathcal{E}[\mathbb{1}_A]$).

Corollary 4.5.8. *Assume $\alpha = \beta = 1$. We set $N = \lfloor e^{ct} \rfloor$ for some $c \in \left(\frac{2}{5}, 1\right)$, and $x_0 = I^{-1}(c) = \sqrt{1-(1-c)^2}$. Then we have*

$$\lim_{t\to\infty} \mathcal{P}\left(\frac{\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} - tx_0}{t^{1/3}d} \leqslant y\right) = F_{\mathrm{GUE}}(y), \tag{4.68}$$

where

$$d = \frac{\left(2c^2\sqrt{1-c}\right)^{1/3}}{\sqrt{1-(1-c)^2}}.$$

Remark 4.5.9. The condition $c > 2/5$ is equivalent to $x_0 > 4/5$. It is also equivalent to the condition that $\theta < 1/2$ in Theorem 4.5.2. Hence, it is most probably purely technical.

Remark 4.5.10. We expect that Corollary 4.5.8 holds more generally for arbitrary parameters $\alpha, \beta > 0$. One would have the following result:

Let $N = \lfloor e^{ct} \rfloor$ such that there exists $x_0 > \frac{\alpha-\beta}{\alpha+\beta}$ and $\theta_0 > 0$ with $x(\theta_0) = x_0$ and $I(x(\theta_0)) = c$. Then

$$\lim_{t\to\infty} \mathcal{P}\left(\frac{\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} - tx_0}{t^{1/3}\sigma(x_0)/I'(x_0)} \leqslant y\right) = F_{\mathrm{GUE}}(y), \tag{4.69}$$

where $I'(x) = \frac{\mathrm{d}}{\mathrm{d}x}I(x)$.

Remark 4.5.11. The range of the parameter c in Corollary 4.5.8 is a priori $c \in (0,1)$. The reason why the upper bound is precisely 1 is because we are in the $\alpha = \beta = 1$ case. In general, the upper bound is $I(1)$, which is always finite. It is natural that c is bounded. Indeed, we know that for all i, $X_t^{(i)} \leqslant t$ (because the random walk performs ± 1 steps), and for c very large there exists some i such that $X_t^{(i)} = t$ with high probability. Hence, one expects that for c large enough, the maximum $\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\}$ is exactly t with a probability going to 1 as t goes to infinity, and there cannot be random fluctuations in that case.

If one considers N simple symmetric random walks (corresponding to the annealed model), the threshold is $\log(2)$ (i.e. for $c > \log(2)$, $(1-(1/2)^t)^N \to 0$ and for $c < \log(2)$, $(1-(1/2)^t)^N \to 1$). One can calculate the large deviations rate function I^a for the simple random walk [1] and check that $I^a(1) = \log(2)$.

1. By Crámer's Theorem, it is the Legendre transform of $z \mapsto \log\left(\frac{e^{-z}+e^{z}}{2}\right)$. One finds

$$I^a(x) = \begin{cases} \frac{1}{2}\big((1+x)\log(1+x) + (1-x)\log(1-x)\big) & \text{for } x \in [-1,1], \\ +\infty & \text{else.} \end{cases}$$

Proof of Corollary 4.5.8. This proof relies on Theorem 4.5.2 which deals only with $\alpha = \beta = 1$. However, this type of deduction would also hold in the general parameter case, and we write the proof using general form expressions. From Theorem 4.5.2, we have that writing

$$\log(\mathsf{P}(X_t > xt)) = -I(x)t + t^{1/3}\sigma(x)\chi_t, \tag{4.70}$$

then χ_t weakly converges to the Tracy-Widom GUE distribution, provided x can be written $x = x(\theta)$ with $0 < \theta < 1/2$. For any realization of the environment, we have on the one hand

$$\mathsf{P}\left(\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} \leqslant xt\right) = \Big(1 - \mathsf{P}(X_t > xt)\Big)^{\lfloor e^{ct}\rfloor} = \exp\Big(\lfloor e^{ct}\rfloor \log\big(1 - \mathsf{P}(X_t > xt)\big)\Big).$$

On the other hand, setting $x = x_0 + \frac{t^{-2/3}\sigma(x_0)y}{I'(x_0)}$, we have that

$$\mathcal{P}\left(\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} \leqslant xt\right) = \mathcal{P}\left(\frac{\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} - tx_0}{t^{1/3}\sigma(x_0)/I'(x_0)} \leqslant y\right). \tag{4.71}$$

By Taylor expansion, we have as t goes to infinity

$$I(x) = I(x_0) + t^{-2/3}\sigma(x_0)y + \mathcal{O}(t^{-4/3}),$$

and

$$\sigma(x) = \sigma(x_0) + t^{-2/3}\frac{\sigma'(x_0)\sigma(x_0)y}{I'(x_0)} + \mathcal{O}(t^{-4/3}).$$

Hence, the R.H.S. of (4.70) is approximated by

$$-I(x)t + t^{1/3}\sigma(x)\chi_t = -I(x_0)t + t^{1/3}\sigma(x_0)(\chi_t - y) + \mathcal{O}(t^{-1/3}) + \mathcal{O}(t^{-1/3}\chi_t). \tag{4.72}$$

Choosing x_0 such that $I(x_0) = c$, we have

$$\begin{aligned}
\mathcal{P}\left(\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\} \leqslant xt\right) &= \mathbb{E}\exp\Big(\lfloor e^{ct}\rfloor \log\big(1 - \mathsf{P}(X_t > xt)\big)\Big)\\
&= \mathbb{E}\exp\Big(- \lfloor e^{ct}\rfloor P(t, xt) + \mathcal{O}\left(e^{ct}P(t, xt)^2\right)\Big)\\
&= \mathbb{E}\exp\Big(e^{t^{1/3}\sigma(x_0)(\chi_t - y) + \mathcal{O}(t^{-1/3}(1+\chi_t))}\\
&\qquad + \mathcal{O}\left(P(t, xt)\right) + \mathcal{O}\left(e^{ct}P(t, xt)^2\right)\Big)
\end{aligned}$$

The second equality relies on Taylor expansion of the logarithm around 1. The third equality is the consequence (4.70) and (4.72). Since χ_t converges in distribution, $t^{-1/3}(1 + \chi_t))$ converges in probability to zero by Slutsky's theorem. Hence, the term $\mathcal{O}(t^{-1/3}(1+\chi_t))$ inside the exponential converges in probability to zero. Recalling that $I(x_0) = c$, we have

$$\begin{aligned}
P(t, xt)^2 &= \exp\left(2\log\left(P(t, xt)\right)\right)\\
&= \exp\Big(2\Big(- ct + \mathcal{O}(t^{1/3}\chi_t)\Big)\Big)\\
&= \exp\Big(- 2ct + 2t^{2/3}\mathcal{O}(t^{-1/3}\chi_t)\Big),
\end{aligned}$$

and since $\mathcal{O}(t^{-1/3}(1+\chi_t))$ converges to zero in probability, we have that $P(t,xt)^2$ is smaller that $e^{-\frac{3}{2}ct}$ with probability going to 1 as t goes to infinity, so that the term $\mathcal{O}\left(e^{ct}P(t,xt)^2\right)$ can be neglected. One can bound similarly $\mathcal{O}(P(t,xt))$ by $e^{-\frac{1}{2}ct}$ with probability going to 1. Thus,

$$\begin{aligned}
\lim_{t\to\infty}\mathcal{P}\left(\frac{\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\}-tx_0}{t^{1/3}\sigma(x_0)/I'(x_0)}\leqslant y\right) &= \lim_{t\to\infty}\mathcal{P}\left(\max_{i=1,\dots,\lfloor e^{ct}\rfloor}\left\{X_t^{(i)}\right\}\leqslant xt\right)\\
&= \lim_{t\to\infty}\mathbb{P}(\chi_t\leqslant y)\\
&= F_{\mathrm{GUE}}(y).
\end{aligned}$$

In the case $\alpha=\beta=1$, we have seen that $I(x)=1-\sqrt{1-x^2}$ so that $x_0=\sqrt{1-(1-c)^2}$. Moreover, using (4.47),

$$\sigma(x_0)/I'(x_0)=d=\frac{\left(2c^2\sqrt{1-c}\right)^{1/3}}{x}=\frac{\left(2c^2\sqrt{1-c}\right)^{1/3}}{\sqrt{1-(1-c)^2}},$$

as in the statement of Corollary 4.5.8. Finally, we have that $I(x(1/2))=2/5$ and $x((1/2))=4/5$, so that the hypothesis of Corollary 4.5.8 match with that of Theorem 4.5.2. $\square$

In order to put Corollary 4.5.8 in the perspective of extreme value statistics, recall that if $(G_i)_i$ for $i=1,\dots,\lfloor e^{ct}\rfloor$ is a sequence of independent Gaussian centred random variables of variance 1, then we have ([Gal87, Section 2.3.2]) the weak convergence

$$\sqrt{2ct}\max_{i=1,\dots,e^{ct}}\{G_i\}+\frac{1}{2}\log(t)+\log(4\pi\sqrt{c})\Longrightarrow\mathcal{G},$$

where $\mathcal{G}$ is a Gumbel random variable with cumulative distribution function $\exp(-e^{-x})$.

For the Beta-RWRE with general $\alpha,\beta>0$ parameters, the variables $X_t^{(i)}$ have mean $\frac{\alpha-\beta}{\alpha+\beta}t$ with variance $\mathcal{O}(t)$ (see Proposition 4.5.12 (1) and (2)). Let us note

$$R_t^{(i)}:=\frac{X_t^{(i)}-\frac{\alpha-\beta}{\alpha+\beta}t}{\sqrt{t}}.$$

We know that $R_t^{(i)}$ converges weakly to the Gaussian distribution by the central limit theorem. Moreover, conditionally on the environment, $R_t^{(i)}$ converges weakly to the Gaussian distribution (It is proved in [RAS05], see the discussion in Section 4.1.6). However, if we let the environment vary, the variables $R_t^{(i)}$ are not independent since the random walks all share the same environment.

The next proposition characterizes the covariance structure of the family $(X_t^{(i)})_{i\geqslant 1}$. We state the result for any parameters $\alpha,\beta>0$.

Proposition 4.5.12. *1. For all $i\geqslant 1$, we have $\mathcal{E}\left[X_t^{(i)}\right]=t\frac{\alpha-\beta}{\alpha+\beta}$.*

2. For all $i\geqslant 1$, we have $\mathcal{E}\left[\left(X_t^{(i)}\right)^2\right]=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2t^2+\frac{4\alpha\beta}{(\alpha+\beta)^2}t$.

3. *For all $i \neq j \geqslant 1$, we have*

$$\mathcal{E}\left[X_t^{(i)} X_t^{(j)}\right] = \left(t\frac{\alpha-\beta}{\alpha+\beta}\right)^2 + \frac{4\alpha\beta\sum_{s=0}^{t-1}\mathcal{P}(X_s^{(i)} = X_s^{(j)})}{(\alpha+\beta)^2(\alpha+\beta+1)}. \tag{4.73}$$

4. *For two random variables X and Y measurable with respect to $\mathcal{P}$, we denote their correlation coefficient as*

$$\rho(X,Y) = \frac{\mathcal{E}[XY]}{\sqrt{\mathcal{E}[X^2]\mathcal{E}[Y^2]}}.$$

For all $i \neq j \geqslant 1$, the correlation coefficient $\rho(X_t^{(i)}, X_t^{(j)})$ equals $1/(\alpha+\beta+1)$ times the $\mathcal{E}$-expected proportion of overlap between the walks $X_t^{(i)}$ and $X_t^{(j)}$, up to time t.

Proof. The points (1) and (2) are trivial since X_t is actually a simple random walk if we do not condition on the environment. In any case, let us explain each case explicitly.

1. Let us write $\Delta_t = X_{t+1} - X_t$. Then $X_t = \sum_{i=0}^{t-1}\Delta_i$. Δ_i is a random variable that takes the value 1 with probability $\mathbb{E}[B]$ and the value -1 with probability $\mathbb{E}[1-B]$ for some $Beta(\alpha,\beta)$ random variable B. We find that $\mathcal{E}\left[\Delta_t\right] = \frac{\alpha-\beta}{\alpha+\beta}$, and

$$\mathcal{E}\left[X_t\right] = \sum_{i=1}^{t}\mathcal{E}\left[\Delta_t\right] = t\frac{\alpha-\beta}{\alpha+\beta}.$$

2. We have

$$\mathcal{E}\left[(X_t)^2\right] = \mathcal{E}\left[\sum_{i=1}^{t}\Delta_i\sum_{j=1}^{t}\Delta_j\right].$$

For $i \neq j$, $\mathcal{E}\left[\Delta_i\Delta_j\right] = \mathcal{E}\left[\Delta_i\right]\mathcal{E}\left[\Delta_j\right]$, and since Δ_i equals plus or minus one, $\mathcal{E}\left[(\Delta_i)^2\right] = 1$. Hence,

$$\mathcal{E}\left[(X_t)^2\right] = t(t-1)\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 + t = \left(t\frac{\alpha-\beta}{\alpha+\beta}\right)^2 + t\frac{4\alpha\beta}{(\alpha+\beta)^2}.$$

3. Let us write $\Delta_t^{(i)} = X_{t+1}^{(i)} - X_t^{(i)}$ and $\Delta_t^{(j)} = X_{t+1}^{(j)} - X_t^{(j)}$. We have

$$\mathcal{E}\left[X_t^{(i)}X_t^{(j)}\right] = \mathcal{E}\left[\sum_{n=0}^{t-1}\Delta_n^{(i)}\sum_{m=0}^{t-1}\Delta_m^{(j)}\right].$$

For $n \neq m$, since the increments and the environments corresponding to different times are independent,

$$\mathcal{E}\left[\Delta_n^{(i)}\Delta_m^{(j)}\right] = \mathcal{E}\left[\Delta_n^{(i)}\right]\mathcal{E}\left[\Delta_m^{(j)}\right] = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2.$$

However, $\mathcal{E}\left[\Delta_n^{(i)}\Delta_n^{(j)}\right]$ depends on whether $X_n^{(i)} = X_n^{(j)}$ or not. More precisely,

$$\mathcal{E}\left[\Delta_n^{(i)}\Delta_n^{(j)}\middle| X_n^{(i)} \neq X_n^{(j)}\right] = \mathcal{E}\left[\Delta_n^{(i)}\right]\mathcal{E}\left[\Delta_m^{(j)}\right] = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2,$$

and

$$\mathcal{E}\left[\Delta_n^{(i)}\Delta_n^{(j)}\middle|X_n^{(i)}=X_n^{(j)}\right]=\mathbb{E}\left[\mathsf{E}\left[\Delta_n^{(i)}\right]\mathsf{E}\left[\Delta_n^{(j)}\right]\middle|X_n^{(i)}=X_n^{(j)}\right]=\mathbb{E}\left[(2B-1)^2\right],$$

for some $Beta(\alpha,\beta)$ random variable B. This yields

$$\mathcal{E}\left[\Delta_n^{(i)}\Delta_n^{(j)}\right]=\mathcal{P}(X_n^{(i)}\neq X_n^{(j)})\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2+\mathcal{P}(X_n^{(i)}=X_n^{(j)})\mathbb{E}\left[(2B-1)^2\right].$$

Using $\mathbb{E}[B^2]=\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$, we find that

$$\mathcal{E}\left[X_t^{(i)}X_t^{(j)}\right]=t^2\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2+\left(\sum_{s=0}^{t-1}\mathcal{P}\left(X_s^{(i)}=X_s^{(j)}\right)\right)\frac{4\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

4. The $\mathcal{E}$-expected proportion of overlap between the walks $X_t^{(i)}$ and $X_t^{(j)}$ up to time t is

$$\frac{1}{t}\mathcal{E}\left[\sum_{s=0}^{t-1}\mathbb{1}_{X_s^{(i)}=X_s^{(j)}}\right]=\frac{1}{t}\sum_{s=0}^{t-1}\mathcal{P}(X_s^{(i)}=X_s^{(j)}).$$

Hence, the point (4) is a direct consequence of (1), (2) and (3).

□

One can precisely describe the behaviour of $\sum_{s=0}^{t-1}\mathcal{P}(X_s^{(i)}=X_s^{(j)})$. For simplicity, we restrict the study to the case where the random walks have no drift, that is $\alpha=\beta$.

Proposition 4.5.13. *Consider $(X_t^{(1)})_{t\in\mathbb{Z}_{\geqslant 0}}$ and $(X_t^{(2)})_{t\in\mathbb{Z}_{\geqslant 0}}$ two Beta-RWRE drawn independently in the same environment with parameters $\alpha=\beta$. Then*

$$\sqrt{t}\ \cdot\ \mathcal{P}\left(X_t^{(1)}=X_t^{(2)}\right)\xrightarrow[t\to\infty]{}\frac{2\alpha+1}{2\alpha}\frac{1}{\sqrt{\pi}},$$

and consequently

$$\sqrt{t}\ \cdot\ \mathcal{E}\left[\frac{X_t^{(i)}}{\sqrt{t}}\frac{X_t^{(j)}}{\sqrt{t}}\right]\xrightarrow[t\to\infty]{}\frac{1}{\alpha\sqrt{\pi}}.$$

Proof. First, notice that $\left(X_t^{(1)}-X_t^{(2)}\right)_{t\geqslant 0}$ is a random walk. Let $Y_t:=X_t^{(1)}-X_t^{(2)}$. The transitions probabilities depend on whether $Y_t=0$. If $Y_t=0$, then

$$Y_{t+1}-Y_t=\begin{cases}+2 & \text{with probability } \mathbb{E}\big[B(1-B)\big]\\ 0 & \text{with probability } \mathbb{E}\big[B^2+(1-B)^2\big]\\ -2 & \text{with probability } \mathbb{E}\big[B(1-B)\big]\end{cases}$$

where B is a $Beta(\alpha,\alpha)$ random variable. If $Y_t\neq 0$, then

$$Y_{t+1}-Y_t=\begin{cases}+2 & \text{with probability } 1/4\\ 0 & \text{with probability } 1/2\\ -2 & \text{with probability } 1/4\end{cases}$$

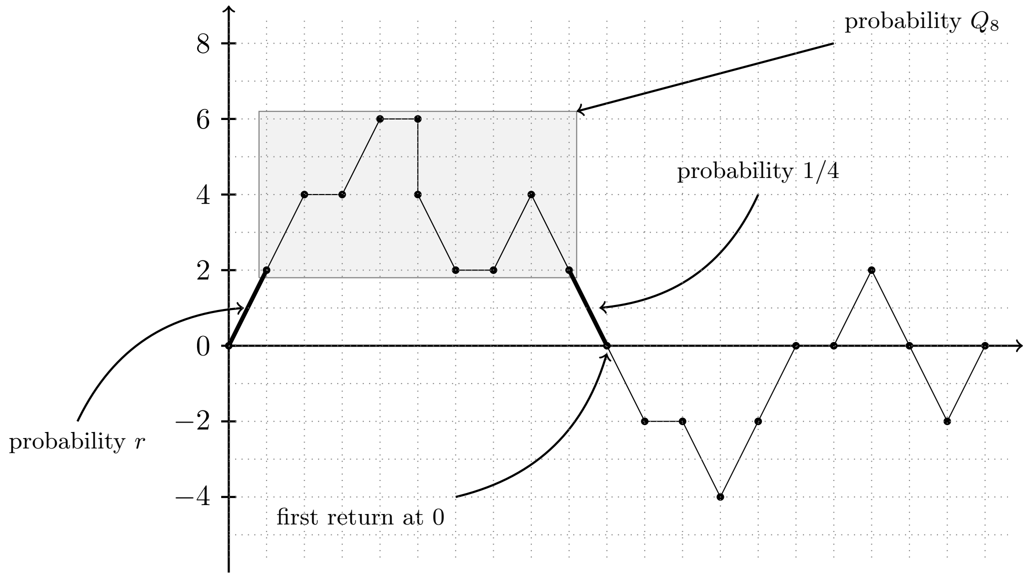

Figure 4.7: A possible trajectory of the random walk Y_t is decomposed to explain the recurrence (4.74). The trajectory in the gray box has the same probability as that of the auxiliary random walk.

In the following, we denote $r = \mathbb{E}\big[B(1-B)\big] = \frac{\alpha}{4\alpha+2}$. We also denote $P_t := \mathcal{P}\left(Y_t = 0\right)$ which is the quantity that we want to approximate.

We introduce an auxiliary random walk starting from 0 and having transitions

$$\begin{cases} +2 & \text{with probability } 1/4, \\ 0 & \text{with probability } 1/2, \\ -2 & \text{with probability } 1/4. \end{cases}$$

We denote by Q_t the probability for the auxiliary random walk to arrive at zero at time t and stay in the non-negative region between times 0 and t.

By conditioning on the first return in zero of the random walk $(Y_t)_t$, we claim that for $t \geqslant 2$,

$$P_t = (1-2r)P_{t-1} + 2\sum_{i=2}^{t} r\frac{1}{4}Q_{i-2}P_{t-i}. \tag{4.74}$$

Let us explain more precisely equation (4.74) (see Figure 4.7):

- The term $(1-2r)P_{t-1}$ corresponds to the case when the first return at zero occur at time 1.
- The factor 2 in front of the sum in (4.74) accounts for the fact that the walk can stay either in the positive, or in the negative region before the first return in zero, with equal probability.
- The factor r is the probability that $Y_1 = 2$ (which is also the probability that $Y_1 = -2$).
- The factor 1/4 is the probability of the last step before the first return at zero.

By conditioning on the first return at zero of the auxiliary random walk, one can see that Q_t verifies the recurrence

$$Q_t = \frac{1}{2}Q_{t-1} + \sum_{i=2}^{t} \frac{1}{16} Q_{i-2}Q_{t-i} \text{ for } t \geqslant 2.$$

This implies that if $Q(z) = \sum_{n\geqslant 0} Q_n z^n$ is the generating function of the sequence $(Q_n)_n$, then

$$Q(z) - 1 - 1/2z = 1/2z(Q(z) - 1) + 1/16z^2 Q(z)^2.$$

This yields

$$Q(z) = \frac{8 - 4z - 8\sqrt{1-z}}{z^2}.$$

Now, let us denote $G(z) = \sum_{n\geqslant 0} P_n z^n$ the generating function of the sequence $(P_n)_n$. The recurrence (4.74) implies that

$$G(z) - 1 - (1-2r)z = (1-2r)z(G(z) - 1) + 2r(1/4)G(z)Q(z).$$

This yields

$$G(z) = \frac{1}{1 + z(4r-1) + 4r(\sqrt{1-z} - 1)}.$$

The function $G(z)$ is analytic in the unit open disk, and can be developed in series around 0 with radius of convergence 1. The nature of its singularities on the unit circle gives the leading order for the asymptotic behaviour of its series coefficients. As $z \to 1$ (for $z \in \mathbb{C} \setminus D$ where D is the cone $D = \{z : |\arg(z-1)| < \epsilon\}$, for some $\epsilon > 0$ arbitrarily small, and taking the branch cut of $\sqrt{1-z}$ along $\mathbb{R}_{\geqslant 1}$),

$$G(z) \sim \frac{1}{4r\sqrt{1-z}},$$

where $\sim$ means that the ratio of the two sides tends to 1 as $z \to 1$ and z belongs to the domain described above. We deduce (from e.g. [FS09, Corollary VI.1]) that

$$P_t \sim \frac{1}{4r}\frac{1}{\sqrt{\pi t}}.$$

This clearly implies that

$$\frac{\sum_{s=0}^{t-1} P_s}{\sqrt{t}} \xrightarrow[t\to\infty]{} \frac{1}{2r\sqrt{\pi}}.$$

Since $r = \frac{\alpha}{4\alpha+2}$ and using (4.73), we get

$$\sqrt{t}\mathcal{E}\left[\frac{X_t^{(i)}}{\sqrt{t}}\frac{X_t^{(j)}}{\sqrt{t}}\right] \xrightarrow[t\to\infty]{} \frac{1}{\alpha\sqrt{\pi}}.$$

$\square$

Comparison to correlated Gaussian variables

Consider for simplicity only the case $\alpha = \beta$. We denote as before $R_t^{(i)} = X_t^{(i)}/\sqrt{t}$. As already mentioned in Section 4.1.6, $R_t^{(i)}$ converges weakly as t goes to infinity to the Gaussian distribution $\mathcal{N}(0,1)$ (whether we condition on the environment or not). It is tempting to ask if the same limit theorem for the maximum holds when one replaces the $R_t^{(i)}$ by the corresponding limiting collection of Gaussian random variables (it would correspond to taking first the limit when t goes to infinity and then study the maximum as N goes to infinity). The theory of extreme value statistics provides a negative answer.

Let $\Sigma_N(\lambda)$ be the matrix of size N

$$\Sigma_N(\lambda) := \begin{pmatrix} 1 & \frac{\lambda}{\sqrt{\log(N)}} & \cdots & \frac{\lambda}{\sqrt{\log(N)}} \\ \frac{\lambda}{\sqrt{\log(N)}} & 1 & & \vdots \\ \vdots & & \ddots & \frac{\lambda}{\sqrt{\log(N)}} \\ \frac{\lambda}{\sqrt{\log(N)}} & \cdots & \frac{\lambda}{\sqrt{\log(N)}} & 1 \end{pmatrix},$$

where $\lambda > 0$ is a parameter. If we set $N = \lfloor e^{ct} \rfloor$, and look at the maximum of the sequence $\{R_t^{(i)}\}_{1\leqslant i\leqslant N}$ as t goes to infinity, the correlation matrix of the sequence is asymptotically $\Sigma_N(\lambda)$ with $\lambda = \frac{\sqrt{c/\pi}}{\alpha}$ (cf. Proposition 4.5.13).

Let $G_N := (G^{(1)}, \dots, G^{(N)})$ be a Gaussian vector with covariance matrix $\Sigma_N(\lambda)$ and let denote the maximum $M_N := \max_{i=1,\dots,N}\{G^{(i)}\}$. Theorem 3.8.1 in [Gal87] implies that we have the convergence in distribution

$$\frac{M_N - \sqrt{2\log(N)} + \lambda\sqrt{2}}{\left(\lambda^{-1}\sqrt{\log(N)}\right)^{-1/2}} \Longrightarrow \mathcal{N}(0,1).$$

In particular, we have the convergence in probability of $M_N/\sqrt{\log(N)}$ to $\sqrt{2}$.

Thus, we have seen that the maximum of $\left(R_t^{(i)}\right)_{1\leqslant i\leqslant N}$ and the maximum of $\left(G^{(i)}\right)_{1\leqslant i\leqslant N}$ obey very different limit theorems: both the scales and the limiting laws are different.

Remark 4.5.14. By Corollary 4.5.8, we have the convergence in probability

$$\frac{\max_{i=1,\dots,N}\{R^{(i)}_{\log(N)/c}\}}{\sqrt{\log(N)}} \xrightarrow[N\to\infty]{\mathcal{P}} \frac{x_0}{\sqrt{c}},$$

where $c = I(x_0)$. Since for any α and $\beta = \alpha$, $I''(0) = 1$, we notice that when $x_0 \to 0$, the approximation at the first order coincide with the Gaussian case. To substantiate this parallel, one must extend to the full parameter range $\alpha, \beta > 0$ and $0 < c < 1$ in Corollary 4.5.8 byond $\alpha = \beta = 1$ and $c > 2/5$ (see also Remark 4.5.9).

Remark 4.5.15. It is clear that the sequence $\left(X_t^{(i)}\right)_{1\leqslant i\leqslant N}$ is exchangeable. There exist general results for maxima of exchangeable sequences. In some cases, one can prove that the maximum, properly renormalized, converges to a mixture of one of the classical extreme laws (see in [Gal87] the discussion in Section 3.2 and the results of Section 3.6). However, it seems that our particular setting does not fit into this theory.

4.6 Asymptotic analysis of the Bernoulli-Exponential FPP

4.6.1 Statement of the result

We investigate the behaviour of the first passage time $T(n, \kappa n)$ when n goes to infinity, for some slope $\kappa > \frac{a}{b}$. When $\kappa = \frac{a}{b}$, the first passage time $T(n, \kappa n)$ should go to zero. The case $\kappa < \frac{a}{b}$ is similar with $\kappa > \frac{a}{b}$ by symmetry.

As in Theorem 4.5.2, we parametrize the slope κ by a parameter θ (which turns out to be the position of the critical point in the asymptotic analysis). Let

$$\kappa(\theta) := \frac{\dfrac{1}{\theta^2} - \dfrac{1}{(a+\theta)^2}}{\dfrac{1}{(a+\theta)^2} - \dfrac{1}{(a+b+\theta)^2}}, \tag{4.75}$$

$$\tau(\theta) := \frac{1}{a+\theta} - \frac{1}{\theta} + \kappa(\theta)\left(\frac{1}{a+\theta} - \frac{1}{a+b+\theta}\right) = \frac{a(a+b)}{\theta^2(2a+b+2\theta)}, \tag{4.76}$$

and

$$\rho(\theta) := \left[\frac{1}{\theta^3} - \frac{1}{(a+\theta)^3} + \kappa(\theta)\left(\frac{1}{(a+b+\theta)^3} - \frac{1}{(a+\theta)^3}\right)\right]^{1/3}. \tag{4.77}$$

When θ ranges from 0 to $+\infty$, $\kappa(\theta)$ ranges from $+\infty$ to a/b and $\tau(\theta)$ ranges from $+\infty$ to 0.

Theorem 4.6.1. *We have that for any $\theta > 0$ and parameters $a, b > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T\big(n, \kappa(\theta)n\big) - \tau(\theta)n}{\rho(\theta)n^{1/3}} \leqslant y\right) = F_{\mathrm{GUE}}(y).$$

By Theorem 4.1.18, we have a Fredholm determinant representation for the probability

$$\mathbb{P}\Big(T\big(n, \kappa(\theta)n\big) > r\Big).$$

We set $r = \tau(\theta)n + \rho(\theta)n^{1/3}y$. Thus, we have that

$$\mathbb{P}\Big(T\big(n, \kappa(\theta)n\big) > \tau(\theta)n + \rho(\theta)n^{1/3}y\Big) = \det(I - K_r^{\mathrm{FPP}})_{\mathbb{L}^2(C_0')},$$

where

$$K_r^{\mathrm{FPP}}(u, u') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \exp\Big(n(H(u+s) - H(u)) + \rho(\theta)n^{1/3}ys\Big)\frac{u+s}{u}\frac{\mathrm{d}s}{s(s+u-u')},$$

and

$$H(z) := \tau(\theta)z + \log\left(\frac{z}{a+z}\right) + \kappa(\theta)\log\left(\frac{a+b+z}{a+z}\right).$$

We have

$$H'(z) = \tau(\theta) + \frac{1}{z} - \frac{1}{a+z} + \kappa(\theta)\left(\frac{1}{a+b+z} - \frac{1}{a+z}\right).$$

and

$$H''(z) = \frac{1}{(a+z)^2} - \frac{1}{z^2} + \kappa(\theta)\left(\frac{1}{(a+z)^2} - \frac{1}{(a+b+z)^2}\right).$$

We can from the expressions for the derivatives of H why it is natural to paramametrize κ, τ and ρ as in (4.75), (4.76) and (4.77): with this choice, we have that $H'(\theta) = H''(\theta) = 0$.

As in Section 4.5, we assume for the moment that the Fredholm determinant contour is a small circle around 0. We do the change of variables $z = u + s$ in the definition of the kernel, so that

$$K_r^{\mathrm{FPP}}(u, u') = \frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \exp\left(n(H(z) - H(u)) + \rho(\theta)n^{1/3}y(z-u)\right)\frac{z}{u}\frac{\mathrm{d}z}{(z-u)(z-u')}. \tag{4.78}$$

Lemma 4.6.2. *For any parameters $a, b > 0$ and $\theta > 0$, we have $H'''(\theta) > 0$.*

Proof. We have

$$H'''(\theta) = \frac{2}{\theta^3} - \frac{2}{(a+\theta)^3} + \frac{\frac{1}{\theta^2} - \frac{1}{(a+\theta)^2}}{\frac{1}{(a+\theta)^2} - \frac{1}{(a+b+\theta)^2}}\left(\frac{2}{(a+b+\theta)^3} - \frac{2}{(a+\theta)^3}\right).$$

Hence we have to show that

$$\left(\frac{2}{\theta^3} - \frac{2}{(a+\theta)^3}\right)\left(\frac{1}{(a+\theta)^2} - \frac{1}{(a+b+\theta)^2}\right) > \left(\frac{2}{(a+\theta)^3} - \frac{2}{(a+b+\theta)^3}\right)\left(\frac{1}{\theta^2} - \frac{1}{(a+\theta)^2}\right). \tag{4.79}$$

By putting each side to the same denominator, we arrive at

$$\begin{aligned} & b(a+b+\theta)(2\theta+2a+b)\left((a+\theta)^3 - \theta^3\right) > a\theta(2\theta+a)\left((a+b+\theta)^3 - (a+\theta)^3\right) \\ \Leftrightarrow\ & ab(a+b)(a+\theta)^2(2a+b+3\theta) > 0. \end{aligned}$$

which clearly holds. □

We notice that given the expression (4.77), $H'''(\theta) = 2\big(\rho(\theta)\big)^3$. By Taylor expansion around θ,

$$H(z) - H(\theta) = \frac{(\rho(\theta)(z-\theta))^3}{3} + \mathcal{O}((z-\theta)^4). \tag{4.80}$$

4.6.2 Deformation of contours

We need to find steep-descent contours for the variables z and u. For the z variable, we choose the contour $\mathcal{D}_\theta = \theta + i\mathbb{R}$ as in Section 4.5. For the u variable, we notice that since we are integrating on a finite contour, it will be enough that $\mathfrak{Re}[H(z)] > \mathfrak{Re}[H(\theta)]$ along the contour (See [TW09] and [BCG14]).

Lemma 4.6.3. *The contour $\mathcal{D}_\theta$ is steep-descent for the function $\mathfrak{Re}[H]$ in the sense that $y \mapsto \mathfrak{Re}[H(\theta+iy)]$ is decreasing for y positive and increasing for y negative.*

Proof. Since $\frac{\mathrm{d}}{\mathrm{d}y}\mathfrak{Re}[H(\theta+iy)] = \mathfrak{Im}[H'(\theta+iy)]$, and using symmetry with respect to the real axis, it is enough to show that for $y>0$, $\mathfrak{Im}[H'(\theta+iy)]>0$. We have

$$\mathfrak{Im}[H'(\theta+iy)] = \frac{y}{(\theta+a)^2+y^2} - \frac{y}{\theta^2+y^2} + \kappa(\theta)\left(\frac{y}{(\theta+a)^2+y^2} - \frac{y}{(\theta+a+b)^2+y^2}\right).$$

Given the expression (4.75) for $\kappa(\theta)$, we have to show that

$$\left(\frac{1}{\theta^2+y^2} - \frac{1}{(\theta+a)^2+y^2}\right)\left(\frac{1}{(a+\theta)^2} - \frac{1}{(a+b+\theta)^2}\right) < \left(\frac{1}{(\theta+a)^2+y^2} - \frac{1}{(\theta+a+b)^2+y^2}\right)\left(\frac{1}{\theta^2} - \frac{1}{(a+\theta)^2}\right). \quad (4.81)$$

Factoring both sides in the inequality (4.81) and cancelling equal factors, one readily sees that it is equivalent to

$$\frac{1}{(\theta^2+y^2)(a+b+\theta)^2} < \frac{1}{((\theta+a+b)^2+y^2)\,\theta^2},$$

which is always satisfied. □

Instead of finding a steep-descent path for the $\mathbb{L}^2$ contour as in Section 4.5, we prove that we can find a contour with suitable properties for asymptotics analysis, following the approach of [BCG14].

Lemma 4.6.4. *There exists a closed continuous path γ in the complex plane, such that*

- *The path γ encloses 0 but not $-a-b$,*
- *The path γ crosses the point θ and departs θ with angles ϕ and $-\phi$, for some $\phi \in (\pi/2, 5\pi/6)$,*
- *Let $B(\theta,\epsilon)$ the ball of radius ϵ centred at θ. For any $\epsilon>0$, there exists $\eta>0$ such that for all $z \in \gamma \setminus B(\theta,\epsilon)$, $\mathfrak{Re}[H(z)] - \mathfrak{Re}[H[\theta]] > \eta$.*

Proof. Since H is analytic away from its singularities, $\mathfrak{Re}[H]$ is a harmonic function. It turns out that the shape of level lines $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)]$ are constrained by the nature and the positions of the singularities of H, and provided H is not too complicated (does not have too many singularities), one can describe these level lines.

We know that level lines can cross only at singularities or critical points. In our case, three branches of the level line $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)]$ cross at θ making angles $\pi/6, \pi/2$ and $5\pi/6$. This can be seen from the Taylor expansion (4.80).

The function H has only three singularities of logarithmic type at 0, $-a$ and $-a+b$. When z goes to infinity, $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)]$ implies $\mathfrak{Re}[\tau(\theta)z] \approx \mathfrak{Re}[H(\theta)]$. Hence, there are two branches that goes to infinity in the direction $\pm\infty i + \mathfrak{Re}[H(\theta)]/\tau(\theta)$. Additionally, one knows by the maximum principle that any closed path formed by portions of level lines must enclose a singularity. Finally, one knows the sign of $\mathfrak{Re}[H(z)]$ around each singularity:

- $\mathfrak{Re}[H(z)] < 0$ for z near 0,
- $\mathfrak{Re}[H(z)] < 0$ for z near $-a-b$,
- $\mathfrak{Re}[H(z)] > 0$ for z near $-a$.

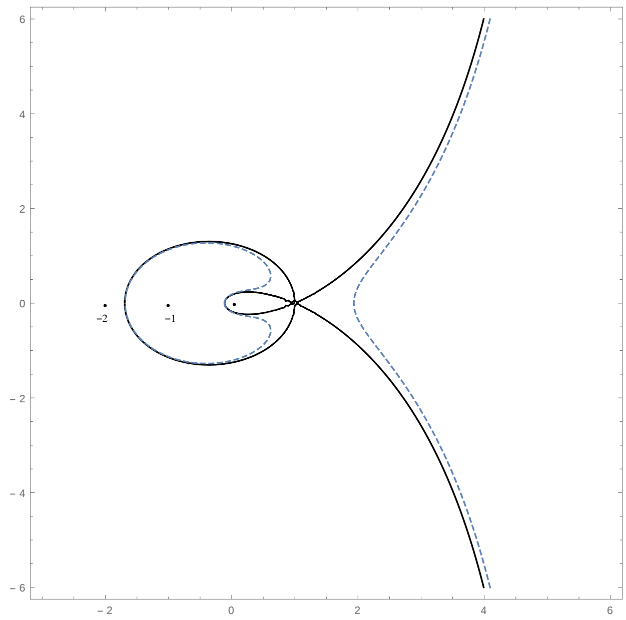

Figure 4.8: The solid lines are contour lines $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)]$ in the case $\theta = a = b = 1$. Dashed lines are contour lines $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)] + 2\eta$ with $\eta = 0.05$.

This is enough to conclude that the level lines of $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)]$ are necessarily as shown in Figure 4.8 (modulo a continuous deformation of the lines that does not cross any singularity). It follows that one can find a path γ having the required properties. It would depart θ with angles $\pm\phi$ with $\phi \in (\pi/2, 5\pi/6)$, and stay between the level lines that depart θ with angles $\pm\pi/2$ and the level lines that departs θ with angles $\pm 5\pi/6$ (For instance, one could follow the level lines of $\mathfrak{Re}[H(z)] = \mathfrak{Re}[H(\theta)] + 2\eta$ outside of a neighbourhood of θ). □

We have the analogue of Proposition 4.5.6.

Proposition 4.6.5. *Let $B(\theta, \epsilon)$ be the ball of radius ϵ centred at θ. We denote by γ^ϵ (resp. $\mathcal{D}_\theta^\epsilon$) the part of the contour γ (resp. $\mathcal{D}_\theta$) inside the ball $B(\theta, \epsilon)$. Then, for any $\epsilon > 0$,*

$$\lim_{t\to\infty} \det(I + K_r^{\mathrm{FPP}})_{\mathbb{L}^2(\mathcal{C}_\theta)} = \lim_{t\to\infty} \det(I + K_{y,\epsilon}^{\mathrm{FPP}})_{\mathbb{L}^2(\gamma^\epsilon)}$$

where $K_{y,\epsilon}^{\mathrm{FPP}}$ is defined by the integral kernel

$$K_{y,\epsilon}^{\mathrm{FPP}}(u, u') = \frac{1}{2i\pi} \int_{\mathcal{D}_\theta^\epsilon} \frac{\pi}{\sin(\pi(z-u))} \exp\Big(t(H(z) - H(u)) - t^{1/3}\rho(\theta) y(z-u)\Big) \frac{\mathrm{d}z}{z - u'}. \tag{4.82}$$

Proof. The proof is similar with the proof of Proposition 4.5.6. The two main differences are

1. The integral defining K_y^{FPP} in (4.78) is an improper integral, which forbids to use dominated convergence.
2. The $\mathbb{L}^2$ contour (i.e. the contour γ) is not steep-descent.

The point (2) is not an issue since in the proof of Proposition 4.5.6, we actually only used the fact that for any $\epsilon > 0$ there exists a constants $C' > 0$ such that $\mathfrak{Re}[h(z)] - \mathfrak{Re}[h(\theta)] > C'$ for $z \in \mathcal{C}_\theta \setminus \mathcal{C}_\theta^\epsilon$. This property is still satisfied by the contour γ.

The point (1) is resolved by bounding the integral over $\mathcal{D}_\theta \setminus \mathcal{D}_\theta^\epsilon$ with the same kind of estimates as in the proof of Theorem 4.1.18. More precisely, one writes

$$\left|\frac{1}{2i\pi}\int_{\theta+i\epsilon}^{\theta+i\infty} \exp\big(n(H(z)-H(u)) + \rho(\theta)n^{1/3}y(z-u)\big)\frac{z}{u}\frac{\mathrm{d}z}{(z-u)(z-u')}\right| <$$
$$\exp\Big(-Cn + n^{1/3}\rho(\theta)y(\theta-u)\Big)\left|\frac{1}{2i\pi}\int_{\theta+i\epsilon}^{\theta+i\infty}\exp\Big(i\rho(\theta)n^{1/3}y\mathfrak{Im}[z]\Big)\frac{z}{u}\frac{\mathrm{d}z}{(z-u)(z-u')}\right|. \tag{4.83}$$

The integral in the R.H.S of (4.83) is an oscillatory integral that can be bounded uniformly in n (actually it goes to zero by Riemann-Lebesgue's lemma) so that it goes to zero when multiplied by $\exp\big(-Cn + n^{1/3}\rho(\theta)y(\theta-u)\big)$. □

The rest of the proof is similar with Section 4.5. One makes the change of variables

$$z = \theta + \tilde{z}n^{-1/3},\ u = \theta + \tilde{u}n^{-1/3}, u' = \theta + \tilde{u}'n^{-1/3}.$$

It is again convenient to deform slightly the contours for u and u' so that the contour for $\tilde{u}$ and $\tilde{u}'$ is $\mathcal{C}^{\epsilon n^{1/3}}$ as in Section 4.5 ($\mathcal{C}^L$ is defined in (4.55)).

Proposition 4.6.6. *We have that*

$$\lim_{t\to\infty}\det(I + K_{y,\epsilon}^{\mathrm{FPP}})_{\mathbb{L}^2(\gamma^\epsilon)} = \det(I - K_y)_{\mathbb{L}^2(\mathcal{C})},$$

where the contour $\mathcal{C}$ is defined in (4.56) and K_y is defined by its integral kernel

$$K_y(w,w') = \frac{1}{2i\pi}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} \frac{\mathrm{d}z}{(z-w')(w-z)}\frac{e^{z^3/3-yz}}{e^{w^3/3-yw}}$$

and the contour for z does not intersect $\mathcal{C}$.

Proof. Identical to the proof of Proposition 4.5.7. □

4.6.3 Limit shape of the percolation cluster for fixed t.

As θ goes to infinity, $\kappa(\theta), \tau(\theta)$ and $\rho(\theta)$ are approximated by

$$\kappa(\theta) = \frac{a}{b} + \frac{3a(a+b)}{2b}\left(\frac{1}{\theta}\right) + \mathcal{O}\left(\frac{1}{\theta}\right)^2,$$
$$\tau(\theta) = \frac{1}{2}a(a+b)\left(\frac{1}{\theta}\right)^3 + \mathcal{O}\left(\frac{1}{\theta}\right)^4,$$
$$\sigma(\theta) = \left(\frac{3}{2}a(a+b)\right)^{1/3}\left(\frac{1}{\theta}\right)^{5/3}.$$

On the other hand, we have from Theorem 4.6.1 the convergence in distribution

$$\frac{T(n,\kappa(\theta)n) - \tau(\theta)n}{\rho(\theta)n^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

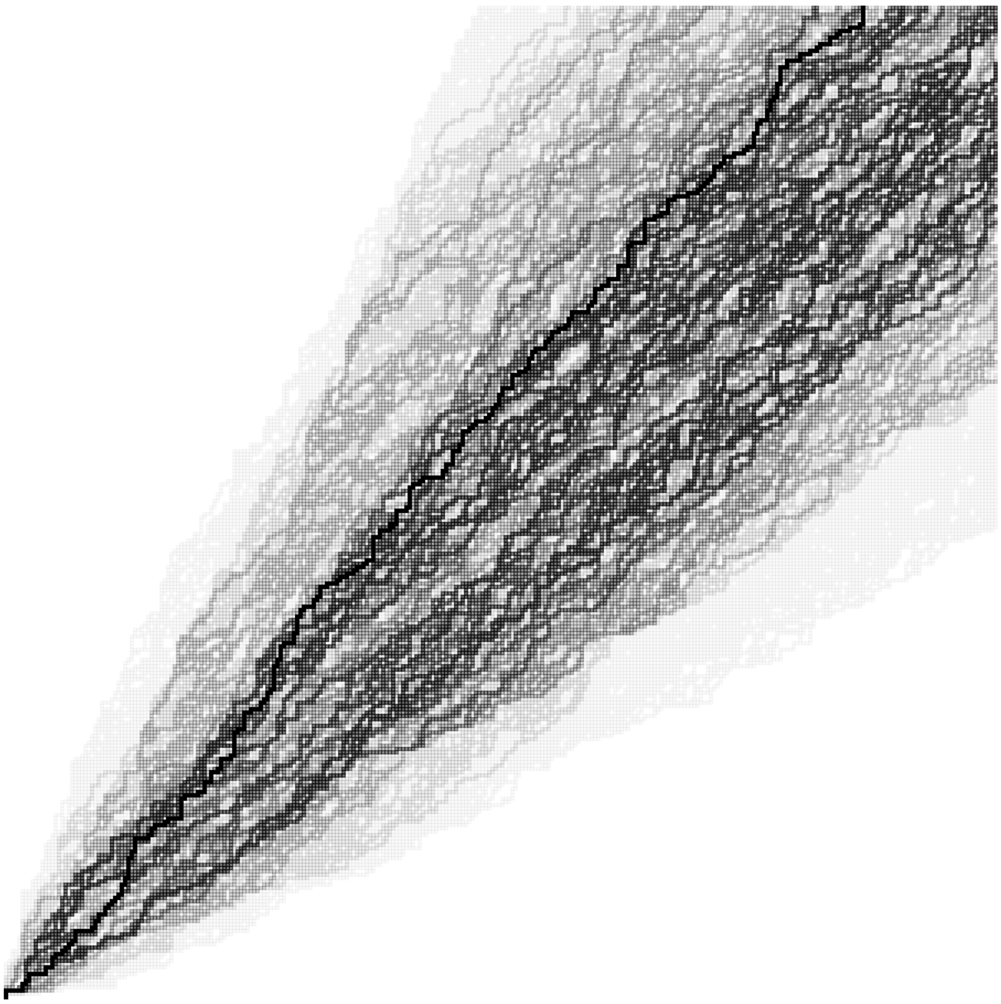

Figure 4.9: Percolation set in the Bernoulli-FPP model at different times for parameters $a = b = 1$. The different shades of gray corresponds to times 0, 0.1, 0.2, 0.3, 0.4, 0.6, 1 and 4. Although it seems on the picture that the convex envelope of the percolation cluster at time $t = 4$ is asymptotically a cone, this is an effect due to the relatively small size of the grid (300×300), and it is not true asymptotically: $n = 300$ is not enough to discriminate between cn and $c'n^{2/3}$ (see Section 4.6.3).

where $\mathcal{L}_{GUE}$ is the GUE Tracy-Widom distribution.

Scaling θ as $n^{1/3}$ suggests a limit theorem for the shape of the convex envelope of the percolation cluster after a fixed time. Of course, there is a non-rigorous interchange of limits here, and one should use the Fredholm determinant representation in order to make this rigorous (we do not include this here).

Let us set $\theta = n^{1/3}$. Then

$$\kappa(\theta)n = \frac{a}{b}n + \frac{3a(a+b)}{2b}n^{2/3} + \mathcal{O}(n^{1/3})$$

and

$$\tau(\theta)n = \frac{1}{2}a(a+b) + \mathcal{O}(n^{-1/3}).$$

This suggests that the border of the percolation cluster at time $\frac{1}{2}a(a+b)$ is asymptotically at a distance $\frac{3a(a+b)}{2b}n^{2/3}$ from the point $\frac{a}{b}n$ (See Figure 4.9). The fact that $\rho(\theta)n^{1/3} = \mathcal{O}(n^{-2/9})$ suggests an anomalous scaling for the fluctuations of the border of the percolation cluster. We leave this for future consideration.

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